Multi-step predictors for direct data-driven analysis and control of nonlinear systems through kernelized velocity forms ⋆

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Abstract: We propose kernel-based approaches for the construction of a single-step and multi-step predictor of the velocity form of nonlinear (NL) systems, which describes the time-difference dynamics of the corresponding NL system and admits a highly structured representation. The predictors in turn allow to formulate completely data-driven representations of the velocity form. The kernel-based formulation that we derive, inherently respects the structured quasi-linear and specific time-dependent relationship of the velocity form. This results in an efficient multi-step predictor for the velocity form and hence for the original NL system, which can be used for direct data-driven analysis and control with global stability and performance guarantees.

Keywords: Direct data-driven control, kernel methods, nonlinear systems, velocity forms.

1. INTRODUCTION AND MOTIVATION

Ever-increasing demands for better performance result in growing complexity for control problems in engineering. This rise in complexity is (partly) due to the fact that these demands translate into larger operating ranges of the inherently NL physical systems. This requires precisely addressing the NL aspects of the behavior of the underlying systems to be able to design a controller that meets the performance demands. Hence, modeling such NL behaviors is crucial, but often becomes cumbersome or even impossible with first-principles modeling. While data-based techniques provide an alternative, there is currently no mature NL identification-for-control theory. This makes it generally hard to decide on (i) which part of the NL behavior is essential to be modeled for control design, and (ii) how the uncertainty of the estimated model influences the subsequent control synthesis. These problems gave rise to the field of direct data-driven control, which focuses on the design of controllers *directly* from data. In the past years, a plethora of systematic methods have been developed that accomplish direct data-based controller design for *linear time-invariant* (LTI) systems (Markovsky and Dörfler, 2021, van Waarde et al., 2023), and recently methods for NL systems have been developed as well. However, despite the promising approaches based on, e.g., online linearizations or polynomial bases (Berberich et al., 2022, De Persis and Tesi, 2023), an analogous result for *general* NL systems that can provide *global* stability and performance guarantees has not been achieved yet.

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The key idea presented in this abstract is to formulate a data-driven characterization of the finite-horizon behavior of the *velocity form* of a general NL system. The velocity form is a representation form of a NL system that describes its *time-difference* dynamics, see, e.g., Koelewijn (2023). The advantage of using this form is that stability and performance properties of the velocity form translate into equilibrium-*independent* properties of the original NL system, i.e., with 'classic' stability analysis of the velocity form, we guarantee *global* stability of the NL system. Hence, achieving a data-driven representation of the finite-horizon behavior of the velocity form results in data-driven analysis and control applications for NL systems that can provide global guarantees.

2. VELOCITY FORMS OF NL SYSTEMS

We consider NL systems that can be represented in the following input-output (IO) form:

$$y(k) = f(y(k-1), \dots, y(k-\ell), u(k), \dots, u(k-\ell)), \quad (1)$$

with $y(k) \in \mathbb{R}^p$, $u(k) \in \mathbb{R}^m$, and $\ell > 0$. We assume that f is continuously differentiable. Let $\eta(k) := \mathrm{col}(y(k-1), \dots, y(k-\ell), u(k), \dots, u(k-\ell))$ such that $y(k) = f(\eta(k))$. The velocity form of (1) is obtained by analyzing the dynamics involving $\Delta u(k) := u(k) - u(k-1)$ and $\Delta y(k) := y(k) - y(k-1)$. Then, through the *fundamental theorem of calculus* and defining w_k as $\eta(k) \cup \eta(k-1)$, i.e, $w_k = \mathrm{col}\big(y(k-1), \dots, y(k-\ell-1), u(k), \dots, u(k-\ell-1)\big)$, the velocity form is given as follows:

$$\Delta y(k) = \sum_{i=1}^{\ell} \mathfrak{a}_i(w_k)\Delta y(k-i) + \sum_{j=0}^{\ell} \mathfrak{b}_j(w_k)\Delta u(k-j), \quad (2)$$

where the coefficient functions $\mathfrak{a}_i(\cdot), \mathfrak{b}_j(\cdot)$ are:

$$\mathfrak{a}_i(w_k) := \int_0^1 \tfrac{\partial f}{\partial y(k-i)}(\xi(k,\lambda))\mathrm{d}\lambda,\ i = 1,\dots,\ell, \tag{3a}$$

$$\mathfrak{b}_j(w_k) := \int_0^1 \tfrac{\partial f}{\partial u(k-j)}(\xi(k,\lambda))\mathrm{d}\lambda,\ j = 0,\dots,\ell, \tag{3b}$$

with $\xi(k,\lambda) := \eta(k-1) + \lambda\big(\eta(k) - \eta(k-1)\big)$. Note that this IO representation of the velocity form of (1) is linear in the Δ-signals and this linear, dynamic relationship is varying along the signals in w_k. In some situations (3) can be computed analytically. In our case, however, we instead assume that (3) are parametrized in terms of a basis function expansion, i.e., with basis functions $\psi_s : \mathbb{R}^{n_\mathrm{w}} \to \mathbb{R}$, $s = 0,\dots,n_\psi$ and coefficients $a_{i,s} \in \mathbb{R}^{p\times p}$, $b_{j,s} \in \mathbb{R}^{p\times m}$, we have, with possibly $n_\psi \to \infty$,

$$\mathfrak{a}_i(w_k) = \textstyle\sum_{s=0}^{n_\psi} a_{i,s}\psi_s(w_k),\ \forall i = 1,\dots,\ell, \tag{4a}$$

$$\mathfrak{b}_j(w_k) = \textstyle\sum_{s=0}^{n_\psi} b_{j,s}\psi_s(w_k),\ \forall j = 0,\dots,\ell. \tag{4b}$$

We assume, w.l.o.g., $\psi_0(w_k) = 1$ for any w_k. Let $\psi(w_k) := \mathrm{col}(\psi_1(w_k),\dots,\psi_{n_\psi}(w_k))$. We will now use kernel-based methods to obtain a multi-step predictor for (2), which will serve as a data-driven representation of the finite-horizon behavior of the velocity form of (1).

3. KERNELIZED REPRESENTATIONS

In Huang et al. (2023) a kernelized L-step predictor for (1) has already been derived. This result could readily be applied in our case. The method in Huang et al. (2023), however, does not take the structural dependencies in (2) with (4) into account, making it impossible to, e.g., increase L without requiring a re-estimation step.

In (Verhoek and Tóth, 2024), we show that a multi-step predictor for (2) with (4) has the form:

$$\Delta y_{[1,L]} = \Theta\,\Psi(w_{[1,L]}) \begin{bmatrix} \Delta y_{[1-\ell,0]} \\ \Delta u_{[1-\ell,0]} \\ \Delta u_{[1,L]} \end{bmatrix}, \tag{5}$$

where $w_{[1,L]}$ denotes the trajectory $(w(1),\dots,w(L))$, similar for $u_{[\cdot,\cdot]}, y_{[\cdot,\cdot]}$, Θ collects the coefficient matrices $a_{i,s}, b_{j,s}$, and $\Psi(w_{[1,L]})$ is a highly structured matrix function that collects $\psi(w_1),\dots,\psi(w_L)$. We preserve here the linearity in the Δ-signals, which is highly advantageous for the data-driven analysis and control methods based on the data-based representation of (5).

Given a data-set $\mathcal{D}^{\mathrm{NL}}_{N+1} = (\breve{u}_{[0,N]}, \breve{y}_{[0,N]})$ taken from (1), we can construct $\mathcal{D}^{\Delta}_N = (\Delta\breve{u}_{[1,N]}, \Delta\breve{y}_{[1,N]})$ for (2). Next, Hankel matrices are assembled by using $\mathcal{D}^{\Delta}_N$ and $\mathcal{D}^{\mathrm{NL}}_{N+1}$: $Y_\ell = \mathcal{H}_\ell(\Delta\breve{y}_{[1,N-L]})$, $Y_L = \mathcal{H}_L(\Delta\breve{y}_{[1+\ell,N]})$, $U_\ell = \mathcal{H}_\ell(\Delta\breve{u}_{[1,N-L]})$, $U_L = \mathcal{H}_L(\Delta\breve{u}_{[1+\ell,N]})$, and $W_L = \mathcal{H}_L(\breve{w}_{[1+\ell,N]})$, where $\breve{w}_{[1+\ell,N]}$ is constructed from $\mathcal{D}^{\mathrm{NL}}_{N+1}$ and the Hankel matrix of a signal $w_{[1,N]}$ is as follows:

$$\mathcal{H}_L(w_{[1,N]}) := \begin{bmatrix} \mathrm{col}(w_{[1,L]}) & \mathrm{col}(w_{[2,L+1]}) & \cdots & \mathrm{col}(w_{[N-L+1,N]}) \end{bmatrix}.$$

Note that $Y_\ell,\dots,W_L$ have the same width. Let N_c be the number of columns of $Y_\ell,\dots,W_L$. We then estimate a model of the form (5) through ridge regression, which yields (see Verhoek and Tóth (2024) for details)

$$Y_L = \Lambda\left(\frac{1}{\gamma}I + (*)^\top \begin{bmatrix} \tilde{\Psi}_{1,1} & \dots & \tilde{\Psi}_{1,N_\mathrm{c}} \\ \vdots & \ddots & \vdots \\ \tilde{\Psi}_{N_\mathrm{c},1} & \dots & \tilde{\Psi}_{N_\mathrm{c},N_\mathrm{c}} \end{bmatrix} \begin{bmatrix} X_{[:,1]} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_{[:,N_\mathrm{c}]} \end{bmatrix}\right),$$

with Λ a matrix composed of *Lagrange multipliers*, $(*)$ indicating a symmetric term, and $X_{[:,i]}$ the i^{th} column of the stacked matrix $\begin{bmatrix} Y_\ell^\top & U_\ell^\top & U_L^\top \end{bmatrix}^\top$. The elements of the middle matrix are

$$\tilde{\Psi}_{i,j} = \Psi(\breve{w}_{[i,i+L-1]})^\top \Psi(\breve{w}_{[j,j+L-1]}),$$

which, when solving for the elements of Ψ, can be written as a diagonal matrix, containing products of $\psi(w_p)^\top\psi(w_q)$, $p,q \in \{1+\ell,\dots,N\}$ on its diagonal. These can be seen as the evaluation of the inner product in a *reproducing kernel Hilbert space* $\mathscr{H}$ if $\psi_s \in \mathscr{H}$, and through the rules for kernel construction from kernels, we can substitute for $\tilde{\Psi}_{i,j}$ a structured matrix kernel function $\mathcal{K}(\breve{w}_{[i,i+L-1]}, \breve{w}_{[j,j+L-1]})$ that inherently characterizes the velocity form with the basis expansions in (3). Introduce $K_{i,j} := \mathcal{K}(\breve{w}_{[i,i+L-1]}, \breve{w}_{[j,j+L-1]})$ and $K_i(w_{[1,L]}) := \mathcal{K}(\breve{w}_{[i,i+L-1]}, w_{[1,L]})$. Then, by substituting Λ, we obtain a fully data-driven predictor:

$$\Delta\hat{y}_{[1,L]} = \mathbf{A}_{\mathcal{D}} \begin{bmatrix} K_1(w_{[1,L]}) \\ \vdots \\ K_{N_\mathrm{c}}(w_{[1,L]}) \end{bmatrix} \begin{bmatrix} \Delta y_{[1-\ell,0]} \\ \Delta u_{[1-\ell,0]} \\ \Delta u_{[1,L]} \end{bmatrix}, \tag{6}$$

with $\mathbf{A}_{\mathcal{D}} := Y_L\left(\frac{1}{\gamma}I + (*)^\top \mathbf{K}_{N_\mathrm{c}} \mathbf{X}_{N_\mathrm{c}}\right)^{-1} \mathbf{X}_{N_\mathrm{c}}^\top$ in which $\mathbf{X}_{N_\mathrm{c}} = \mathrm{diag}\left(X_{[:,1]},\dots,X_{[:,N_\mathrm{c}]}\right)$ and $\mathbf{K}_{N_\mathrm{c}} = \begin{bmatrix} K_{1,1} & \dots & K_{1,N_\mathrm{c}} \\ \vdots & \ddots & \vdots \\ K_{N_\mathrm{c},1} & \dots & K_{N_\mathrm{c},N_\mathrm{c}} \end{bmatrix}$. Note that $\mathbf{A}_{\mathcal{D}}$ is fully constructed from $\mathcal{D}^{\mathrm{NL}}_{N+1}$. The final step towards control design of NL systems from data is the embedding of (6) as a *linear parameter-varying* (LPV) description, and apply the existing tools from the data-driven LPV control framework, such as the predictive control results in Verhoek et al. (2024).

In summary, we combine velocity forms, kernelized representations and convex, linear tools from the LPV framework to achieve data-driven analysis and control of NL systems with global stability and performance guarantees. We refer the interested reader to Verhoek and Tóth (2024), where the content of this extended abstract is rigorously formalized and derived.

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