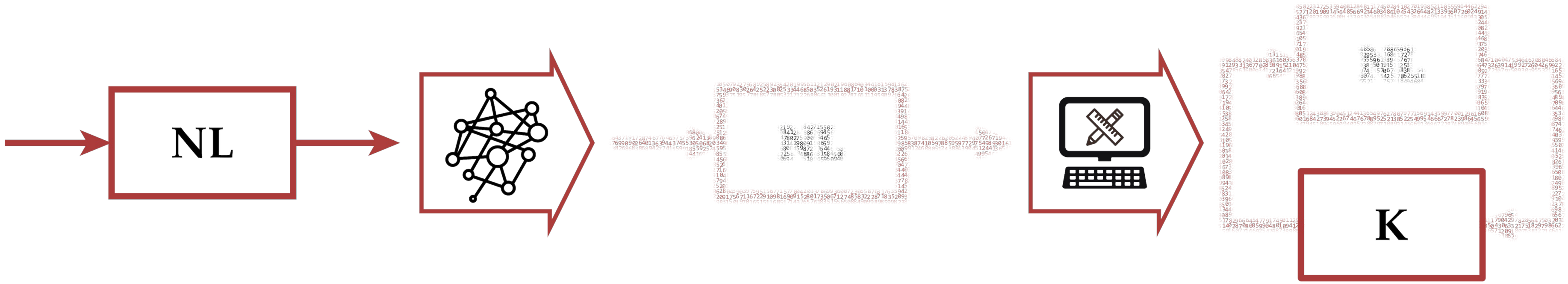


Systematic controller design for nonlinear systems, directly from data

Zardini lab, LIDS @ MIT

June 9, 2025

Chris Verhoek

Control Systems group, Dept. Electrical Engineering, Eindhoven University of Technology, The Netherlands

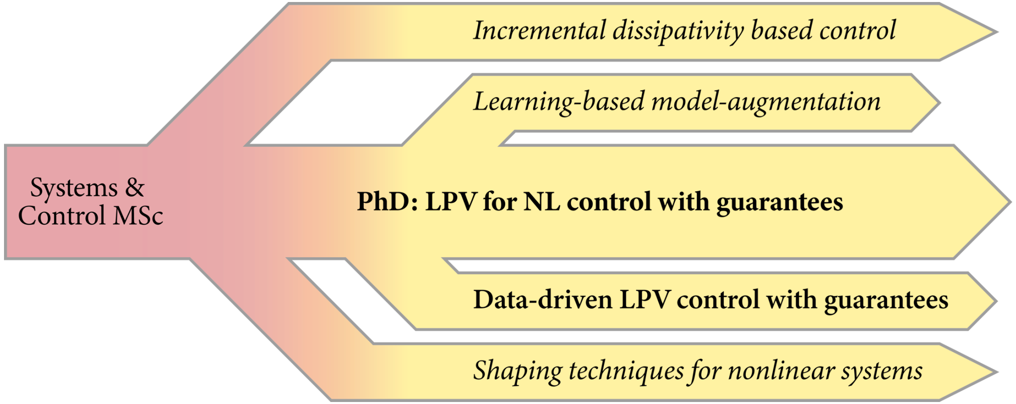
A few things about me

Eindhoven University of Technology (TU/e)

2020: MSc in Systems & Control (*Cum Laude*)

2025: PhD Control Systems group (EE) (*Cum Laude*)

- Roland Tóth & Sofie Haesaert

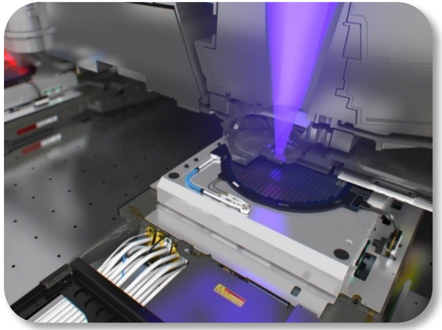


A few things about me

Hiking (multi-day trails)

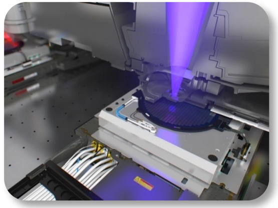
Drumming (jazz)

Motivation



Motivation



Complexity growth

Increased performance demands

- Energy efficiency
- Higher speed & accuracy
- Harsher disturbances

Larger operating range

Motivation

Modeling

$$x^+ = f(x,u)$$
$$y = h(x,u)$$

Simplification

$$x^+ = f(x,u)$$
$$y = h(x,u)$$

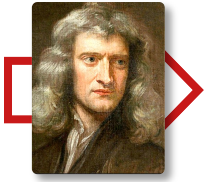
$$x^+ = f(x,u)$$
$$y = h(x,u)$$

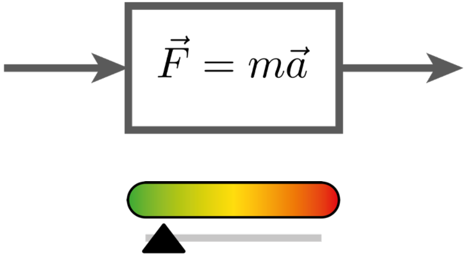
$$\mathbf{K}$$

Analysis & control with guarantees

Motivation

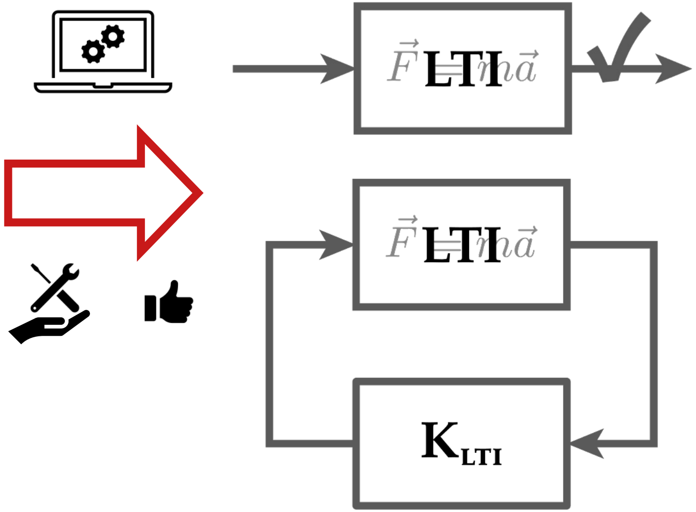
First-Principles Modeling





Linear
Time-Invariant

LTI framework

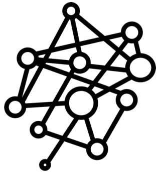


Analysis & control with guarantees

Motivation

Complexity growth → **Modeling**
- Incomplete or even impossible...
- Costly

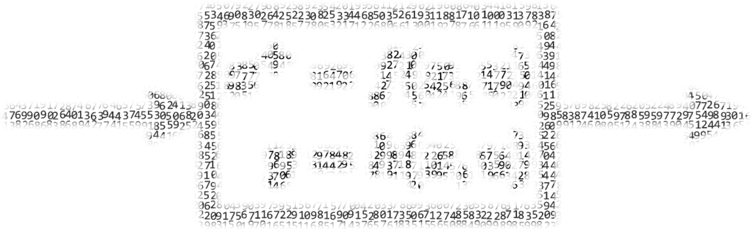
Larger operating range → **Simplification**
- LTI tools insufficient
- Limitation guarantees



Complexity growth

Overcome modeling step

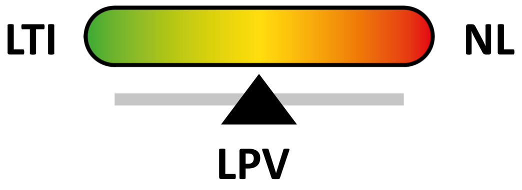
- Use data to describe system behavior
- How to achieve analysis and control directly?



Larger operating range

Address nonlinear behavior

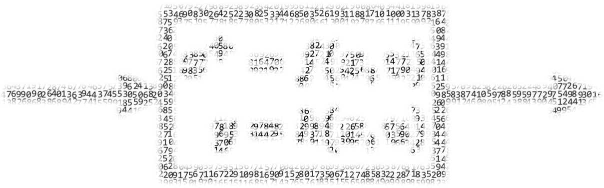
- Keep efficiency of LTI
- Through Linear Parameter-Varying (**LPV**) framework



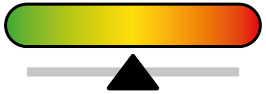
Complexity growth

Larger operating range

Use data to directly describe system behavior!



Address NL behavior through LPV framework!



LPV

Today’s menu: Present a framework that addresses these!

- Systematically represent nonlinear systems from data
- Computationally efficient tools for design
- Give rigorous guarantees

Three main ingredients

Data-driven LPV representations

- Fundamentals in data-driven control paradigm

Data-driven LPV analysis & control tools

- Generalization of LTI tools for design from data

Data-based global guarantees for nonlinear systems

- Efficient handling of nonlinear systems over full operating range

Contents

Data-driven LPV representations

- Background in behavioral LTI data-driven analysis & control
- Behavioral approach for LPV systems
- Data-driven LPV representations

Data-driven LPV analysis & control tools

Data-based global guarantees for nonlinear systems

Direct data-driven control

LTI approaches

- Frequency-domain methods
  - PID tuning [1]
  - Nyquist stability (conservative) [2]
  - Nyquist stability (necessary & sufficient) [3]
  - MIMO stab. through approximation [4]
- Time-domain methods
  - Virtual-feedback reference tuning (VFRT) [5]
  - Non-iterative correlation-based tuning (CbT) [6]
  - Behavioral methods [7,8]
  - Many more …

LPV approaches

- Time-domain approaches
  - VFRT methods [9]
- Frequency-domain
  - Nyquist-based, conservative [10]
- Behavioral [12]

NL approaches

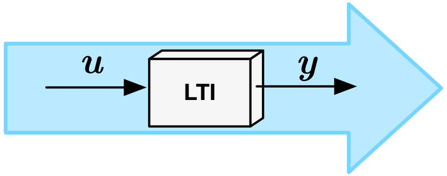
- Sector bounded static nonlinearites [13]
- Behavioral (LTI+, Wiener & Ham.) [14-16]

[1] K. Aström, et al., ECC, 2013
[2] S. Khadraoui, et al., Automatica, 2014
[3] A. Karimi et al., Int. J. Rob. Cont., 2018
[4] A. Karimi et al., Automatica 2017
[5] M. Campi, et al., Automatica, 2002
[6] van Heusden, et al. Int. J. ACDS, 2011
[7] Markovsky, Dörfler, Ann.R. Cont.,2022
[8] van Waarde, et al., TAC, 2023
[9] Formentin et al. Automatica, 2016
[10] Kunze et. al, ECC , 2007
[11] Bloemers et. al., IEEE-LCSS, 2022
[12] Verhoek et.al., IEEE TAC 2022
[13] Nicoletti et al., J. Rob. Cont., 2018
[14] Alsalti ,et. al., IEEE TAC, 2023
[15] Mishra, et. al., ESPC, 2021
[16] Berberich, et. al., ECC, 2021

LTI behavioral theory

Behavioral concept (discrete time)

Linearity



Time axis: $\mathbb{T} = \mathbb{Z}$

Behavior: $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$

- Linearity: $\mathfrak{B}$ is a linear subspace of $\mathbb{W}^{\mathbb{T}}$
- Time invariance: $\mathfrak{B} = q^{\tau}\mathfrak{B}, \forall \tau \in \mathbb{Z}$

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$$

Signal space: $\mathbb{W} = \mathbb{R}^{n_{\mathrm{w}}}$

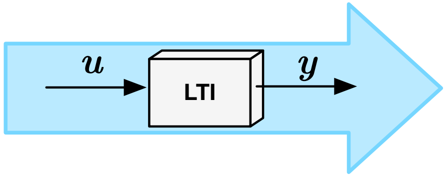
$$w = \mathrm{col}(u, y)$$

[16] Polderman, Willems: Int. to Mathematical Systems Theory: A Behavioral Approach, *Springer* (1998)

LTI behavioral theory

Kernel representation (discrete time)

Linearity



Coefficients: $r_i \in \mathbb{R}^{n \times n_{\mathrm{w}}}$

$$R(q)w = \sum_{i=0}^{n_{\mathrm{r}}} r_i q^i w = 0$$

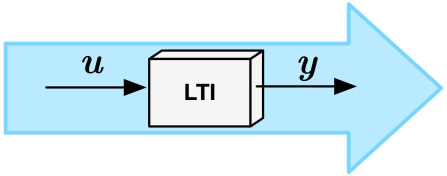
Forward time-shift: $q^i w_k = w_{k+1}$

[16] Polderman, Willems: Int. to Mathematical Systems Theory: A Behavioral Approach, *Springer* (1998)

LTI behavioral theory

Kernel representation (discrete time)

Linearity



$$R(q)w = \sum_{i=0}^{n_\mathrm{r}} r_i q^i w = 0$$

Existence of full row-rank kernel representation

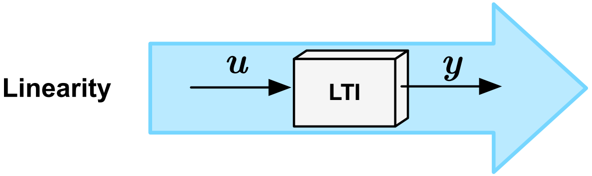
is the representation of the LTI system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ if

$$\mathfrak{B} = \{w \in (\mathbb{R}^{n_\mathrm{w}})^{\mathbb{Z}} \mid R(q)w = 0\}$$

[16] Polderman, Willems: Int. to Mathematical Systems Theory: A Behavioral Approach, *Springer* (1998)

Data-driven LTI behavioral representation

Data-driven representation (discrete time)



$$\mathcal{D}_N = \{\underbrace{u_k^{\mathrm{d}}, y_k^{\mathrm{d}}}_{w_k^{\mathrm{d}}}\}_{k=1}^{N}$$

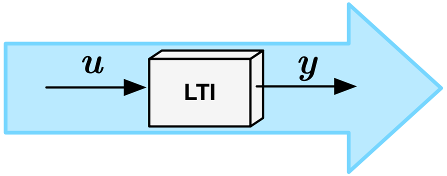
(data dictionary)

[17] Willems et al.: A note on persistency of excitation, *Systems & Control Letters,* (2005).

Data-driven LTI behavioral representation

Data-driven representation (discrete time)

Linearity



Hankel matrix (L-row):

$$\mathcal{H}_L(w^{\mathrm{d}}) = \begin{bmatrix} w_1^{\mathrm{d}} & w_2^{\mathrm{d}} & \cdots & w_{N-L+1}^{\mathrm{d}} \\ w_2^{\mathrm{d}} & w_3^{\mathrm{d}} & \cdots & w_{N-L+2}^{\mathrm{d}} \\ \vdots & \vdots & \ddots & \vdots \\ w_L^{\mathrm{d}} & w_{L+1}^{\mathrm{d}} & \cdots & w_N^{\mathrm{d}} \end{bmatrix}$$

(data dictionary)

Willems' Fundamental Lemma [17]:

$$\mathrm{image}(\mathcal{H}_L(w^{\mathrm{d}})) = \mathfrak{B}|_{[1,L]}$$

if $\mathrm{rank}(\mathcal{H}_L(u^{\mathrm{d}})) = n_{\mathrm{u}}(L + n_{\mathrm{x}})$

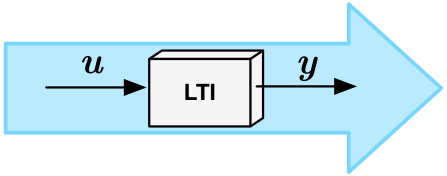
(Persistency of excitation)

[17] Willems et al.: A note on persistency of excitation, *Systems & Control Letters*, (2005).

Data-driven LTI behavioral representation

Data-driven representation (discrete time)

Linearity



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(data dictionary)

Data-driven representation:

$$\exists g \in \mathbb{R}^{N-L+1}$$

$$\begin{bmatrix} \mathcal{H}_L(u^{\mathrm{d}}) \\ \mathcal{H}_L(y^{\mathrm{d}}) \end{bmatrix} g = \begin{bmatrix} \mathrm{col}(\bar{u}_{[1,L]}) \\ \mathrm{col}(\bar{y}_{[1,L]}) \end{bmatrix}$$

$$\Updownarrow$$

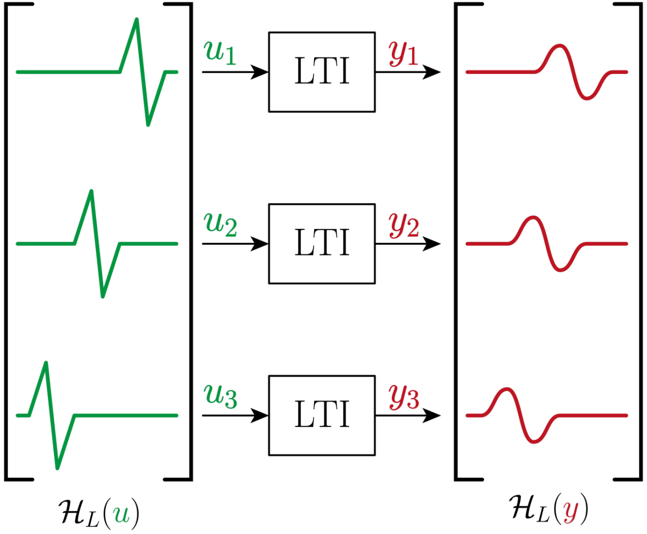
$$\mathrm{col}(\bar{y}_{[1,L]}, \bar{u}_{[1,L]}) \in \mathfrak{B}_{[1,L]}$$

[17] Willems et al.: A note on persistency of excitation, *Systems & Control Letters*, (2005).
[18] Markovsky & Dörfler: Identifiability in the behavioral setting. *IEEE-TAC* (2022)

Data-driven LTI behavioral representation

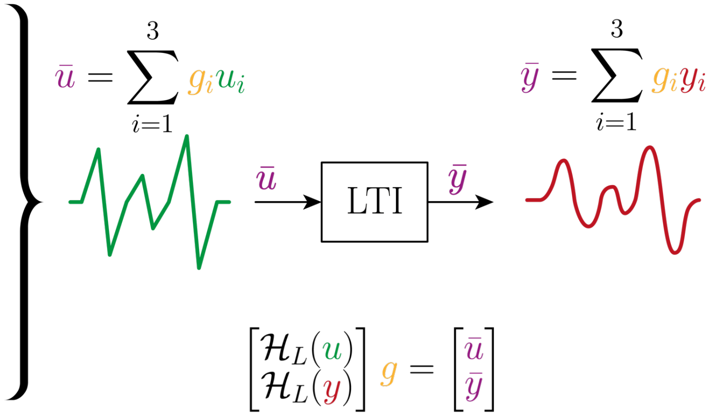
Data-driven representation (discrete time)

Gathered data:



Superposition principle:

$$\bar{u} = \sum_{i=1}^{3} g_i u_i \qquad \bar{y} = \sum_{i=1}^{3} g_i y_i$$



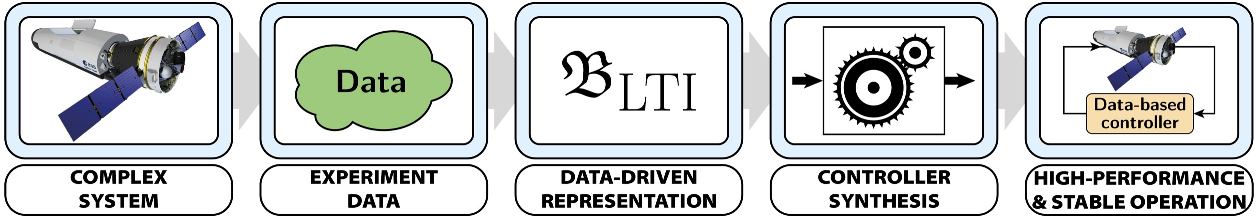
$$\begin{bmatrix} \mathcal{H}_L(u) \\ \mathcal{H}_L(y) \end{bmatrix} g = \begin{bmatrix} \bar{u} \\ \bar{y} \end{bmatrix}$$

(Data-driven representation)

Data-driven LTI behavioral control

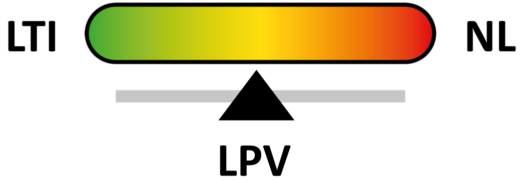
Direct data-driven analysis and control design

- Analysis
  - Simulation (Data spans the full behavior of length L) [7]
  - Stability & performance analysis (dissipativity, quadratic perf., etc.) [19]
- Control
  - Predictive control schemes (e.g., DeePC [8])
  - State-feedback control [7]
  - Noise handling & robustness guarantees [20]



[7] Markovsky, et.al.: Data-driven simulation and control, *Int. Journal of Control,* (2008)
[8] Coulson, et.al.: Data-Enabled Predictive Control: In the Shallows of the DeePC, in *Proc. of the ECC,* (2019)
[19] Romer et al.: One-shot verification of dissipativity properties from input-output data, *Control Systems Letters,* (2019)
[20] Berberich et al.: Data-Driven Model Predictive Control With Stability and Robustness Guarantees, *IEEE TAC,* (2021)

Beyond linear systems?



Linear parameter-varying framework

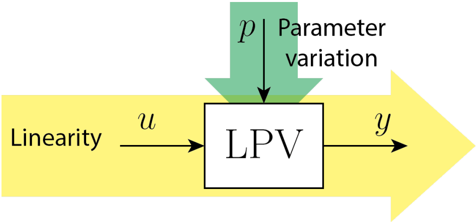
The Engineers' Dream:

How to use "simple" linear control for NL systems with performance guarantees?

Linear parameter-varying framework

Linear systems, varying along a measurable scheduling signal $p \in \mathbb{P}$

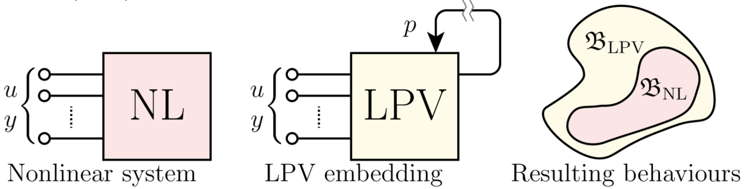
$$\begin{aligned} x_{k+1} &= A(p_k)x_k + B(p_k)u_k \\ y_k &= C(p_k)x_k + D(p_k)u_k \end{aligned}$$



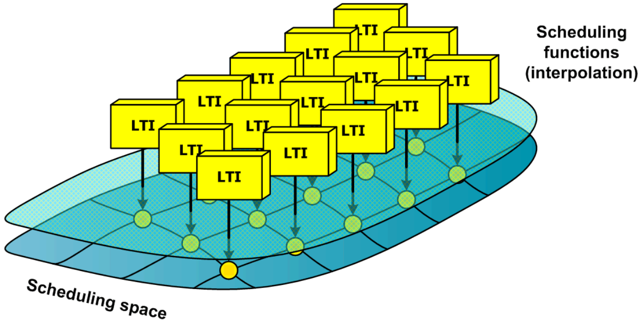
Usage: surrogate models of nonlinear systems (*embedding principle*)

$$\left.\begin{aligned} x_{k+1} &= f(x_k, u_k), \\ y_k &= h(x_k, u_k) \end{aligned}\right\} \xrightarrow[p_k \in \mathbb{P} \subseteq \psi(\mathbb{X},\mathbb{U})]{p_k := \psi(x_k, u_k)} \left\{\begin{aligned} x_{k+1} &= A(p_k)x_k + B(p_k)u_k \\ y_k &= C(p_k)x_k + D(p_k)u_k \end{aligned}\right.$$

Trading a bit of conservatism for linearity!

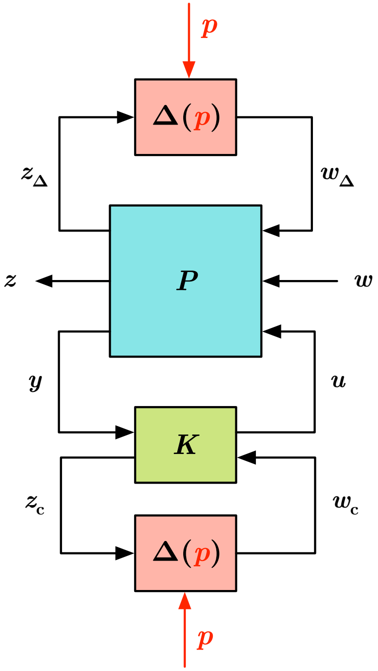


Linear parameter-varying framework

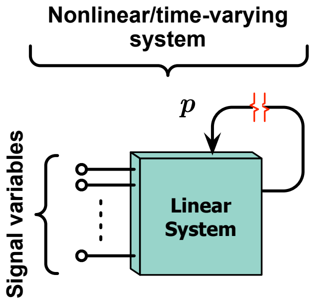


Local approximation principle

Local synthesis:
Gain scheduling
(interpolated LTI control)



Controller Synthesis



Global embedding principle

Global synthesis:
Optimal LPV control
(NL control)

Linear parameter-varying framework

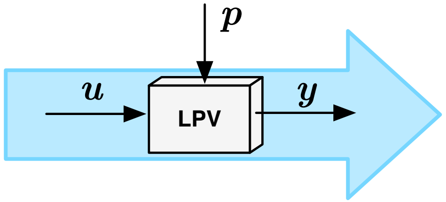
A plethora of success stories via model-based control

Pending question:
How to achieve data-driven LPV control with guarantees?

LPV behavioral theory

Behavioral concept (discrete time)

Linearity



Time axis: $\mathbb{T} = \mathbb{Z}$

Behavior: $\mathfrak{B} \subseteq (\mathbb{W} \times \mathbb{P})^{\mathbb{T}}$

- Linearity: $\mathfrak{B}$ is a linear subspace for some p
- Time invariance: $\mathfrak{B} = q^{\tau}\mathfrak{B}, \forall \tau \in \mathbb{Z}$

$$\Sigma = (\mathbb{T}, \mathbb{P}, \mathbb{W}, \mathfrak{B})$$

Scheduling space: $\mathbb{P} \subseteq \mathbb{R}^{n_\mathrm{p}}$

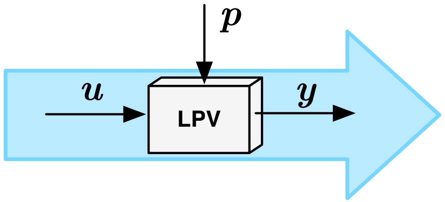
Signal space: $\mathbb{W} = \mathbb{R}^{n_\mathrm{w}}$

$$w = \mathrm{col}(u, y)$$

[21] Tóth: Modeling and Identification of Linear Parameter-Varying Systems, *Springer*, (2010)

LPV behavioral theory

Linearity



Kernel representation (discrete time)

Coefficient functions:

$r_i : \mathbb{R}^{n_\mathrm{p}} \to \mathbb{R}^{n \times n_\mathrm{w}}$

Types (static dep.):

- Affine/linear functions
- Polynomial functions
- Rational functions
- Meromorphic functions

$r(\cdot) = \dfrac{g(\cdot)}{h(\cdot)}$ holomorphic $h \neq 0$

Meromorphic field

$r_i \in \mathcal{R}^{n \times n_\mathrm{w}}$

$$\sum_{i=0}^{n_\mathrm{r}} r_i(p_k) q^i w_k = 0$$

Shift operator: $q^i w_k = w_{k+1}$

Confined in $\mathbb{P} \subseteq \mathbb{R}^{n_\mathrm{p}}$

Signals: $w : \mathbb{Z} \to \mathbb{R}^{n_\mathrm{w}}$

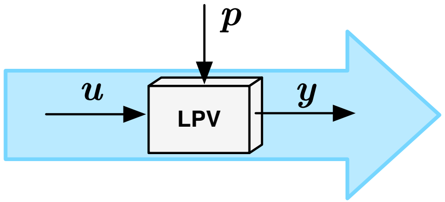
Commutation problem (dynamic dependence):

$$q^i r_i(p_k) w_k \neq r_i(p_k) w_{k+i}$$

[21] Tóth: Modeling and Identification of Linear Parameter-Varying Systems, *Springer*, (2010)

LPV behavioral theory

Linearity



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$$\sum_{i=0}^{n_\mathrm{r}} r_i(p_k) q^i w_k = 0$$

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Confined in

$\mathbb{P} \subseteq \mathbb{R}^{n_\mathrm{p}}$

Signals:

$w : \mathbb{Z} \to \mathbb{R}^{n_\mathrm{w}}$

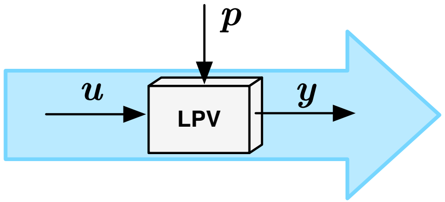
Commutation problem (dynamic dependence):

$$q^i r_i(p_k) w_k = r_i(p_{k+i}) w_{k+i}$$

[21] Tóth: Modeling and Identification of Linear Parameter-Varying Systems, *Springer*, (2010)

LPV behavioral theory

Linearity



Kernel representation (discrete time)

Coefficient functions with finite dynamic dependence

Features:

- Causal
  $r_i(p_k, p_{k-1}, p_{k-2}, \dots)$
- Non-causal
  $r_i(\dots, p_{k+1}, p_k, p_{k-1}, \dots)$

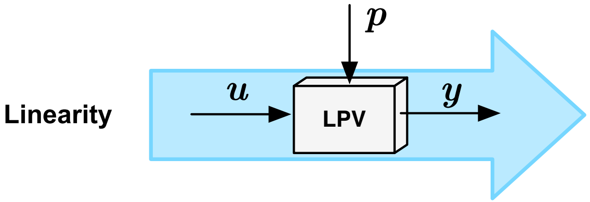
Shorthand for evaluation over *dynamic* dependence

$$\underbrace{\sum_{i=0}^{n_\mathrm{r}} (r_i \diamond p)_k \, q^i}_{R(q)\diamond p} w_k = 0$$

Polynomials over $\mathcal{R}$
$R \in \mathcal{R}[\xi]^{n \times n_\mathrm{w}}$

[21] Tóth: Modeling and Identification of Linear Parameter-Varying Systems, *Springer*, (2010)

LPV behavioral theory



Kernel representation (discrete time)

$$\underbrace{\sum_{i=0}^{n_\mathrm{r}} (r_i \diamond p)_k \, q^i}_{R(q)\diamond p} w_k = 0$$

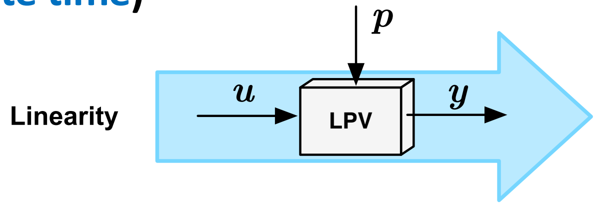
is the representation of the LPV system $\Sigma = (\mathbb{T}, \mathbb{P}, \mathbb{W}, \mathfrak{B})$ if

$$\mathfrak{B} = \{(w, p) \in (\mathbb{R}^{n_\mathrm{w}} \times \mathbb{P})^{\mathbb{Z}} \mid (R(q) \diamond p) w = 0\}$$

[21] Tóth: Modeling and Identification of Linear Parameter-Varying Systems, *Springer,* (2010)

LPV behavioral theory

Behavioral concept (discrete time)



$$\Sigma = (\mathbb{T}, \mathbb{P}, \mathbb{W}, \mathfrak{B})$$

- Projected scheduling behavior:

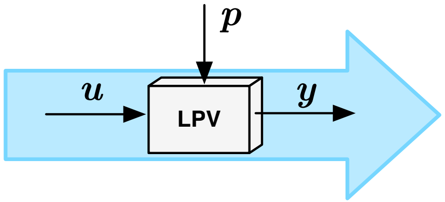
$$\mathfrak{B}_{\mathbb{P}} = \pi_p \mathfrak{B} := \{p \in \mathbb{P}^{\mathbb{T}} \mid \exists w \in \mathbb{W}^{\mathbb{T}} \text{ s.t. } (w, p) \in \mathfrak{B}\}$$

- Projected behavior for a given: $p \in \mathfrak{B}_{\mathbb{P}}$

$$\mathfrak{B}_p = \{w \in \mathbb{W}^{\mathbb{T}} \mid (w, p) \in \mathfrak{B}\}$$

[21] Tóth: Modeling and Identification of Linear Parameter-Varying Systems, *Springer,* (2010)

Data-driven LPV behavioral representation



Data-driven representation (discrete time)

$$\underbrace{\sum_{i=0}^{n_\mathrm{r}} (r_i \diamond p)_k \, q^i}_{R(q)\diamond p} w_k = 0$$

Data-dictionary:

$$\mathcal{D}_N = \{u_k^\mathrm{d}, y_k^\mathrm{d}, p_k^\mathrm{d}\}_{k=1}^N$$

Complex condition
Can we simplify this to an easily *computable* form / representation?

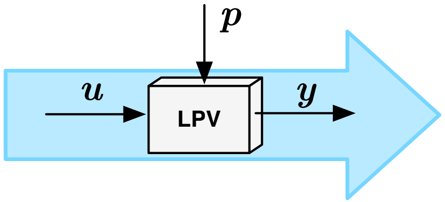
LPV Fundamental Lemma:

$$\mathrm{span}^{\mathrm{col}}_{\mathcal{R},p}(\mathcal{H}_L(w^\mathrm{d})) = \mathfrak{B}_p|_{[1,L]}$$

(Persistency of excitation)
existence of a "unique" R w.r.t $\mathcal{D}_N$.

[22] Verhoek, et. al: Fundamental Lemma for Data-Driven Analysis of Linear Parameter-Varying Systems, *CDC*, (2021)

Data-driven LPV behavioral representation



Data-driven representation (discrete time, simplified case)

Consider the IO form (partitioned kernel rep.):

$$y_k + \sum_{i=1}^{n_\mathrm{a}} a_i(p_{k-i}) y_{k-i} = \sum_{i=1}^{n_\mathrm{b}} b_i(p_{k-i}) u_{k-i}$$

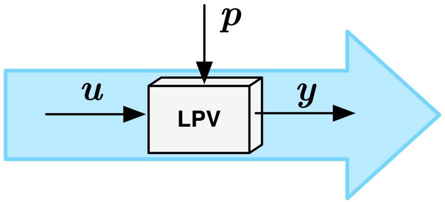
Restricted, but useful subclass of LPV systems

with shifted-affine scheduling dependence:

$$a_i(p_{k-i}) = a_{i,0} + \sum_{j=1}^{n_\mathrm{p}} a_{i,j} p_{j,k-i}, \quad b_i(p_{k-i}) = b_{i,0} + \sum_{j=1}^{n_\mathrm{p}} b_{i,j} p_{j,k-i}$$

[22] Verhoek, et. al: Fundamental Lemma for Data-Driven Analysis of Linear Parameter-Varying Systems, *CDC,* (2021)
[23] Verhoek, et. al: The behavioral approach to data-driven LPV representations, *Accepted for IEEE-TAC,* (2024)

Data-driven LPV behavioral representation



Data-driven representation (discrete time, simplified case)

$$y_k + \sum_{i=1}^{n_\mathrm{a}} a_i \underbrace{(p_{k-i}) y_{k-i}}_{p_{k-i} \otimes y_{k-i}} = \sum_{i=1}^{n_\mathrm{b}} b_i \underbrace{(p_{k-i}) u_{k-i}}_{p_{k-i} \otimes u_{k-i}}$$

Data-dictionary:

$$\mathcal{D}_N = \{u_k^\mathrm{d}, y_k^\mathrm{d}, p_k^\mathrm{d}\}_{k=1}^{N}$$

Data-driven representation:

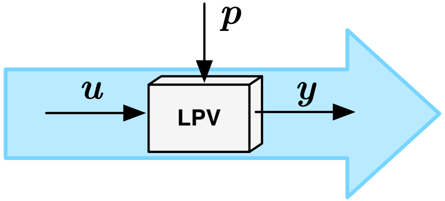
$$\begin{bmatrix} \mathcal{H}_L(u^\mathrm{d}) \\ \mathcal{H}_L(y^\mathrm{d}) \\ \mathcal{H}_L(p^\mathrm{d} \otimes u^\mathrm{d}) - \bar{\mathcal{P}}^{n_\mathrm{u}} \mathcal{H}_L(u^\mathrm{d}) \\ \mathcal{H}_L(p^\mathrm{d} \otimes y^\mathrm{d}) - \bar{\mathcal{P}}^{n_\mathrm{y}} \mathcal{H}_L(y^\mathrm{d}) \end{bmatrix} g = \begin{bmatrix} \mathrm{col}(\bar{u}_{[1,L]}) \\ \mathrm{col}(\bar{y}_{[1,L]}) \\ 0 \\ 0 \end{bmatrix}$$

$$\Updownarrow$$

$$\mathrm{col}(\bar{y}_{[1,L]}, \bar{u}_{[1,L]}, \bar{p}_{[1,L]}) \in \mathfrak{B}_{[1,L]}$$

[22] Verhoek, et. al: Fundamental Lemma for Data-Driven Analysis of Linear Parameter-Varying Systems, *CDC*, (2021)
[23] Verhoek, et. al: The behavioral approach to data-driven LPV representations, *Accepted for IEEE-TAC*, (2024)

Data-driven LPV behavioral representation



Simplified LPV Fundamental Lemma (discrete time)

Given $\mathcal{D}_N = \{u_k^\mathrm{d}, y_k^\mathrm{d}, p_k^\mathrm{d}\}_{k=1}^N$ and let

$$\mathcal{N}_{\bar{p}} := \mathrm{nullspace}\left\{\begin{bmatrix}\mathcal{H}_L(p^\mathrm{d}\otimes u^\mathrm{d}) - \bar{\mathcal{P}}^{n_\mathrm{u}}\mathcal{H}_L(u^\mathrm{d}) \\ \mathcal{H}_L(p^\mathrm{d}\otimes y^\mathrm{d}) - \bar{\mathcal{P}}^{n_\mathrm{y}}\mathcal{H}_L(y^\mathrm{d})\end{bmatrix}\right\}, \quad \mathcal{H}_L(w^\mathrm{d}) := \begin{bmatrix}\mathcal{H}_L(u^\mathrm{d}) \\ \mathcal{H}_L(y^\mathrm{d})\end{bmatrix}$$

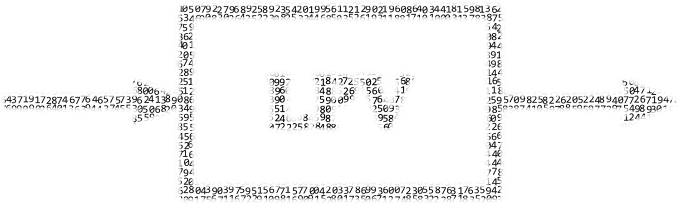
Then, for all scheduling signals $\bar{p} \in \mathfrak{B}_\mathbb{P}$

$$\mathfrak{B}_{\bar{p}}|_{[1,L]} = \mathrm{image}(\mathcal{H}_L(w^\mathrm{d})\mathcal{N}_{\bar{p}}) \iff \mathrm{rank}(\mathcal{H}_L(w^\mathrm{d})\mathcal{N}_{\bar{p}}) = n_\mathrm{x} + n_\mathrm{u}L$$

- The state-feedback case follows the same arguments
- Test for all $\bar{p} \in \mathfrak{B}_\mathbb{P}$ reduces to finite test (see [23])

[23] Verhoek, et. al: The behavioral approach to data-driven LPV representations, *Accepted for IEEE-TAC,* (2024)

Contents



Data-driven LPV representations

Data-driven LPV analysis & control tools

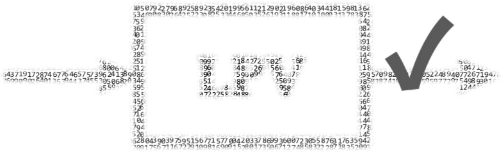
- Direct analysis and control from data
- Computationally efficient tools
- Application in practice

Data-based global guarantees for nonlinear systems

Data-driven LPV analysis & control tools

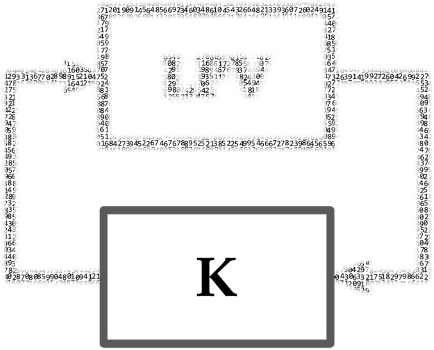
Use representation for developing efficient tools!

- Analysis
  - Stability
  - Performance (dissipativity)

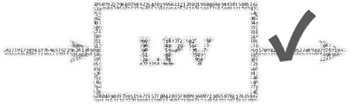


$$\begin{bmatrix} \mathcal{H}_L(u^{\mathrm{d}}) \\ \mathcal{H}_L(y^{\mathrm{d}}) \\ \mathcal{H}_L(p^{\mathrm{d}} \otimes u^{\mathrm{d}}) - \bar{\mathcal{P}}^{n_{\mathrm{u}}}\mathcal{H}_L(u^{\mathrm{d}}) \\ \mathcal{H}_L(p^{\mathrm{d}} \otimes y^{\mathrm{d}}) - \bar{\mathcal{P}}^{n_{\mathrm{y}}}\mathcal{H}_L(y^{\mathrm{d}}) \end{bmatrix} g = \begin{bmatrix} \mathrm{col}(\bar{u}_{[1,L]}) \\ \mathrm{col}(\bar{y}_{[1,L]}) \\ 0 \\ 0 \end{bmatrix}$$

- Control
  - State-feedback/output-feedback
  - Predictive control
  - Trajectory planning



Data-driven LPV analysis & control tools



Analysis tools:

- Simulation directly from data [23]
- Interpolation directly from data [24]
- Dissipativity (performance) analysis directly from data [25]

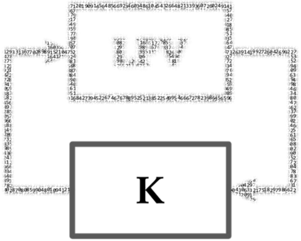
$$\begin{bmatrix} \mu F^\top F & \mu F^\top D_1 \\ \mu D_1{}^\top F & \mu D_1{}^\top D_1 + \Pi_{\mathrm{perf}} \end{bmatrix} - \lambda \begin{bmatrix} I & 0 \\ 0 & D_2 \end{bmatrix}^\top M_{\mathbb{P}} \begin{bmatrix} I & 0 \\ 0 & D_2 \end{bmatrix} \succeq 0$$

Problems are solved as QP/SDP:

- Computationally efficient
- Easy to implement
- Rigorous guarantees!

[23] Verhoek, et. al: The behavioral approach to data-driven LPV representations, *Accepted for IEEE-TAC,* (2024)
[24] Verhoek, et. al: Simulation, interpolation and approximation using data-driven LPV representations, *To appear,* (2025)
[25] Verhoek, et. al: Data-driven Dissipativity Analysis of Linear Parameter-Varying Systems, *IEEE-TAC,* (2023)

Data-driven LPV analysis & control tools



Controller design tools:

- Trajectory planning directly from data [24]
- Predictive control directly from data [26]
- State-feedback synthesis directly from data [27]

$$
\begin{aligned}
\min_{g_k} \quad & \textstyle\sum_{i=0}^{N-1} \ell(\bar{u}_{i|k}, \bar{y}_{i|k}) \\
\text{s.t.} \quad & \begin{bmatrix} \mathcal{H}_N(u^{\mathrm{d}}) \\ \mathcal{H}_N(y^{\mathrm{d}}) \\ \mathcal{H}_N(p^{\mathrm{d}}) \otimes u^{\mathrm{d}} - \bar{\mathcal{P}}^{n_{\mathrm{u}}} \mathcal{H}_N(u^{\mathrm{d}}) \\ \mathcal{H}_N(p^{\mathrm{d}}) \otimes y^{\mathrm{d}} - \bar{\mathcal{P}}^{n_{\mathrm{y}}} \mathcal{H}_N(y^{\mathrm{d}}) \end{bmatrix} g = \begin{bmatrix} \mathrm{col}(\bar{u}_{[1,N]}) \\ \mathrm{col}(\bar{y}_{[1,N]}) \\ 0 \\ 0 \end{bmatrix}, \\
& \bar{u}_{i|k} \in \mathbb{U}, \ \bar{y}_{i|k} \in \mathbb{Y}, \ \forall i \in \mathbb{I}_0^{N-1}, \\
& \begin{bmatrix} u_\tau^{\mathrm{r}} \\ y_\tau^{\mathrm{r}} \end{bmatrix} = \begin{bmatrix} \mathrm{vec}(\bar{u}_{[N-\tau,N-1]|k}) \\ \mathrm{vec}(\bar{y}_{[N-\tau,N-1]|k}) \end{bmatrix},
\end{aligned}
$$

Cost function

Representation

Constraints

Terminal ingred.

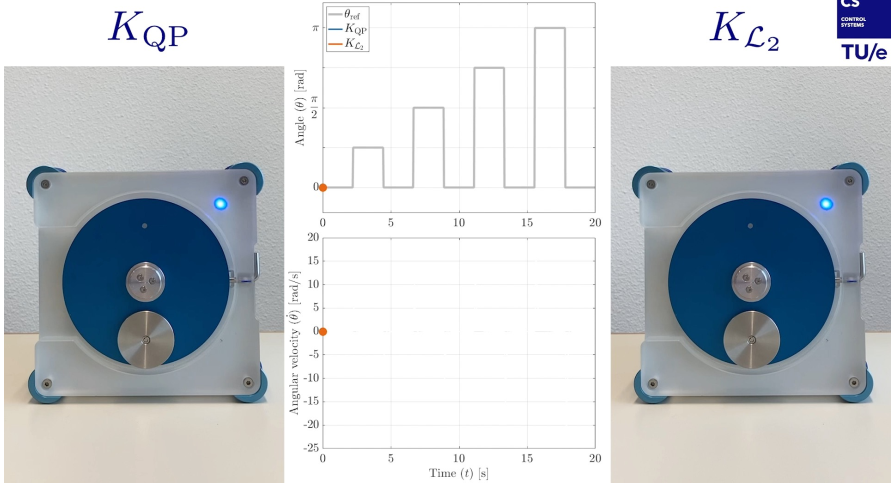
Problems are solved as QP/SDP:

- Computationally efficient
- Easy to implement
- Rigorous guarantees!

[24] Verhoek, et. al: Simulation, interpolation and approximation using data-driven LPV representations, *To appear,* (2025)
[26] Verhoek, et. al: A Linear Parameter-Varying Approach to Data Predictive Control, *Accepted for IEEE-TAC* (2024)
[27] Verhoek, et. al: Direct Data-Driven State-Feedback Control of Linear Parameter-Varying Systems, *Int. J. Rob. & NL Contr.,* (2022)

Data-driven LPV analysis & control tools

Applied in practice (unbalanced disc):

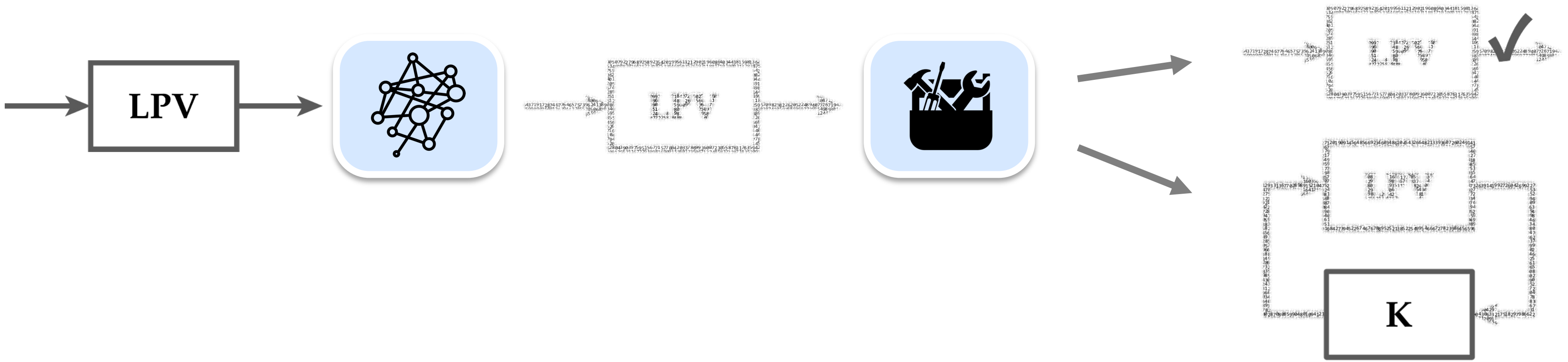


Optimal state-feedback design

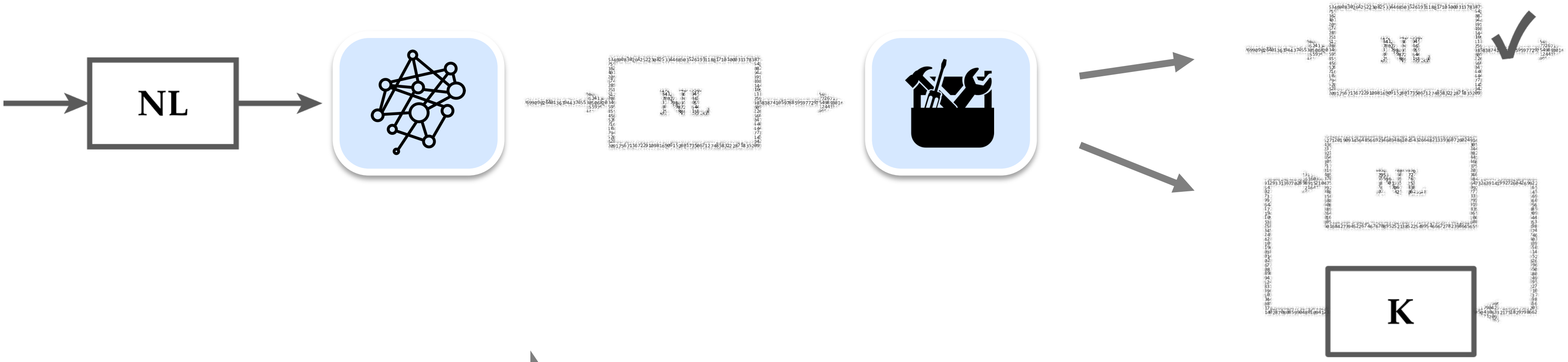
Data-driven advantage
LPV controller synthesized using **7** data-points (70 milliseconds)!

[28] Verhoek, et. al: Direct data-driven LPV control of nonlinear systems: An experimental result, *IFAC WC,* (2023)

Data-driven LPV analysis & control tools

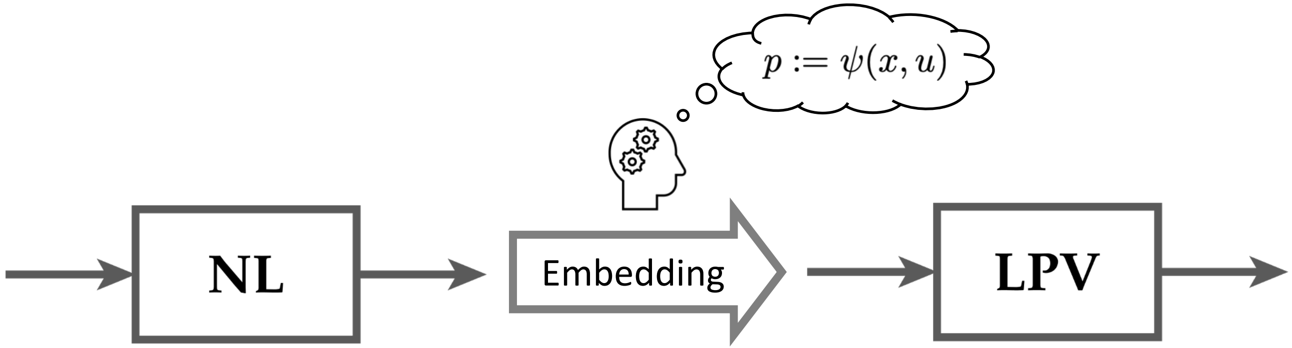


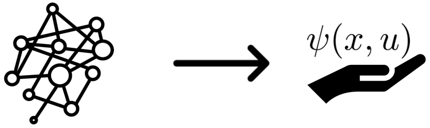
Data-driven LPV analysis & control tools



$$\begin{aligned} x^+ &= f(x,u) \\ y &= h(x,u) \end{aligned} \quad \xrightarrow{\; p := \psi(x,u) \;} \quad \begin{aligned} x^+ &= A(p)x + B(p)u \\ y &= C(p)x + D(p)u \end{aligned}$$

Data-driven LPV analysis & control tools

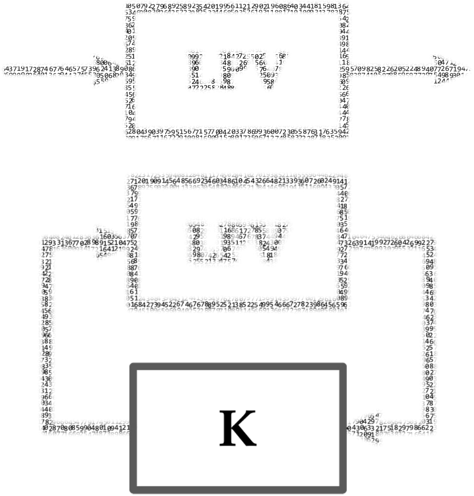




Given the scheduling map

Valid in region around origin

Contents



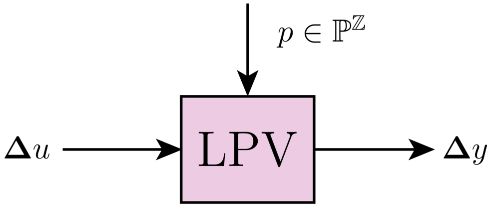
Data-driven LPV representations

Data-driven LPV analysis & control tools

Data-based global guarantees for nonlinear systems

- Learn scheduling map from data
- Achieve global guarantees through velocity form

Learn scheduling map from data



LPV velocity system representation

- Now assume the basis function expansion of scheduling map

$$\begin{aligned} x^+ &= f(x,u) \\ y &= h(x,u) \end{aligned} \quad \xrightarrow{\; p := \psi(x,u) \;} \quad \begin{aligned} x^+ &= A(p)x + B(p)u \\ y &= C(p)x + D(p)u \end{aligned}$$

with

Online measurements

Scheduling map $\psi(\bar{w}) = \sum_{i=1}^{N} \alpha_i \kappa_{w_i}(\bar{w})$

Data-dictionary

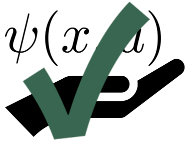
Characterize with data

➢ Learn the basis functions from data with **kernel** methods

[39] Verhoek, Tóth: Kernel-based multi-step predictors for data-driven analysis and control of nonlinear systems through the velocity form, *arXiv:2408.00688*, (2024).

Kernelized data-driven representations

- Employ RKHS and LS-SVM framework
- Existing kernelized data-driven representations → unstructured [40]
- Exploit **linear** structure of LPV form through kernel construction
    - Efficient & flexible
    - Quasi-linear expansion can be achieved for cont. diff. maps
    - Direct representation or scheduling map recovery
    - Earlier presented tools **directly** applicable!

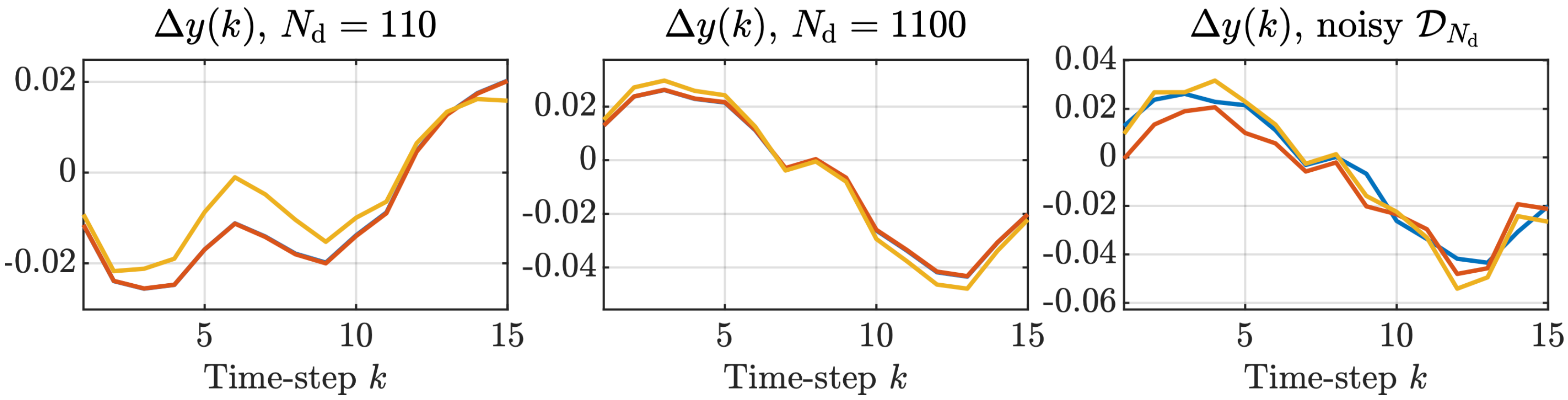


[39] Verhoek, Tóth: Kernel-based multi-step predictors for data-driven analysis and control of nonlinear systems through the velocity form, *arXiv:2408.00688*, (2024).
[40] Huang, Lygeros, Dörfler: Robust and kernelized data-enabled predictive control for nonlinear systems, IEEE-TCST, 2024.

Kernelized data-driven representations

A short simulation example – compare structured/unstructured formulations

- Unbalanced disc system (true)
- L step ahead predictor ($L = 15$)
- Small/large/noisy data set



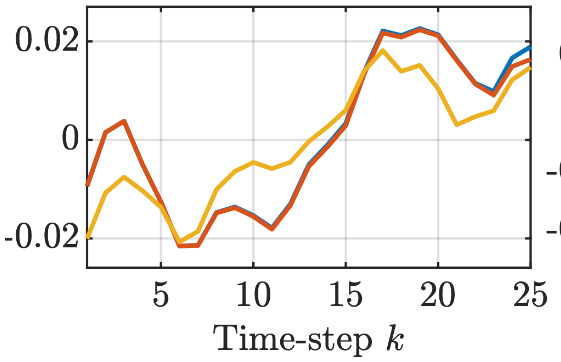
[39] Verhoek, Tóth: Kernel-based multi-step predictors for data-driven analysis and control of nonlinear systems through the velocity form, *arXiv:2408.00688*, (2024).

Kernelized data-driven representations

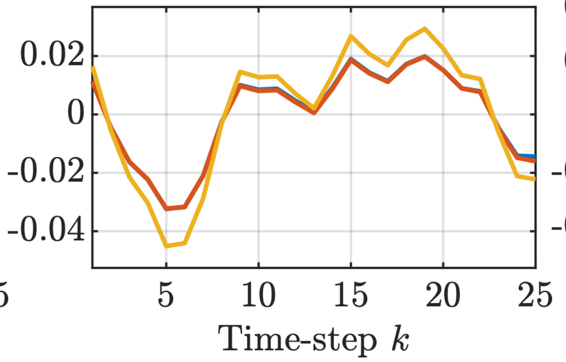
A short simulation example – compare structured/unstructured formulations

- Unbalanced disc system (true)
- L step ahead predictor ($L = 25$), **same kernels/hyperparameters/etc.**
- Small/large/noisy data set

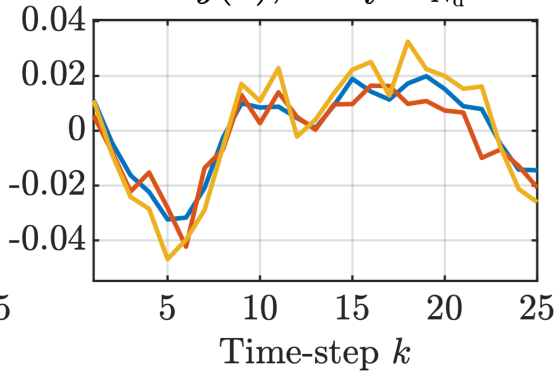
$\Delta y(k),\ N_\mathrm{d} = 110$



$\Delta y(k),\ N_\mathrm{d} = 1100$



$\Delta y(k)$, noisy $\mathcal{D}_{N_\mathrm{d}}$



[39] Verhoek, Tóth: Kernel-based multi-step predictors for data-driven analysis and control of nonlinear systems through the velocity form, *arXiv:2408.00688*, (2024).

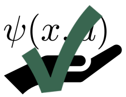
Contents

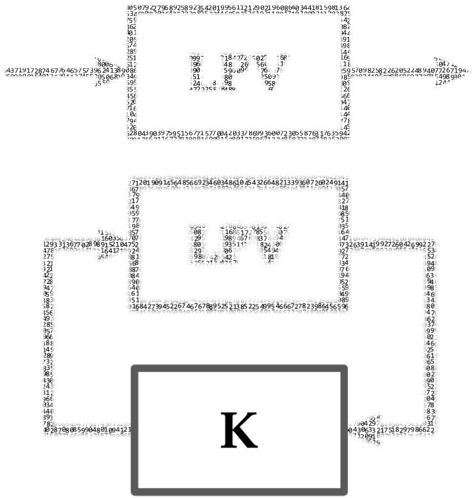
Data-driven LPV representations

Data-driven LPV analysis & control tools

Data-based global guarantees for nonlinear systems

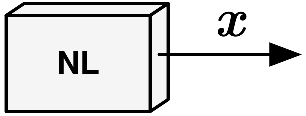
- ~~Learn scheduling map from data~~
- Achieve global guarantees through velocity form





Concept of global stability

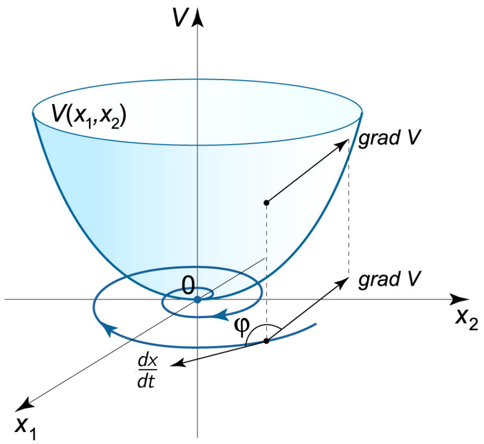
Nonlinear system (autonomous, discrete time)

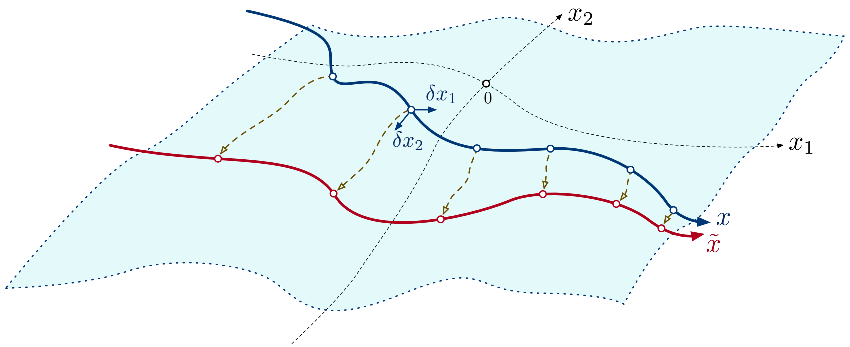


$$x_{k+1} = f(x_k)$$

$f \in \mathcal{C}^1$

Concept of global stability





Lyapunov Stability

Core stability concept in the NL/LPV context

Incremental Stability

Convergence of trajectories (equilibrium free stability)

Can fail in case of tracking!

Closed-loop NL guarantees can be lost.

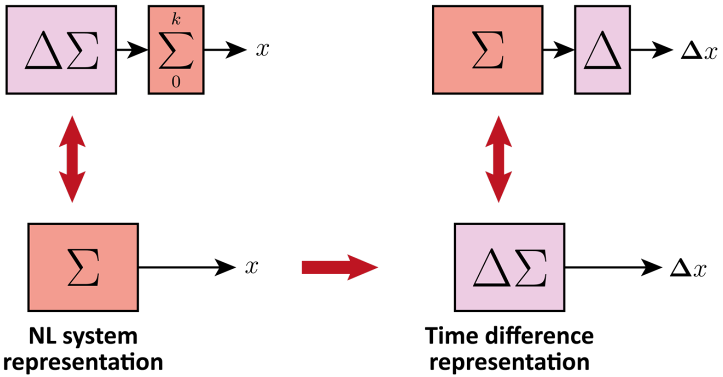
Concept of global stability

- **Krasovskii type of condition**
  - Consider the time difference form

$$\Delta x_{k+1} = f(x_k) - f(x_{k-1}) \qquad \Delta x_0 \in \mathbb{R}^{n_\mathrm{x}}$$

$$\Delta x_{k+1} = \int_0^1 \frac{\partial f}{\partial x}(\breve{x}_k(\lambda))\, d\lambda \cdot \Delta x_k \qquad \breve{x}_k(\lambda) = x_{k-1} + \lambda(x_k - x_{k-1})$$

$$\Delta x_{k+1} = \mathcal{A}(x_k, x_{k-1})\Delta x_k$$

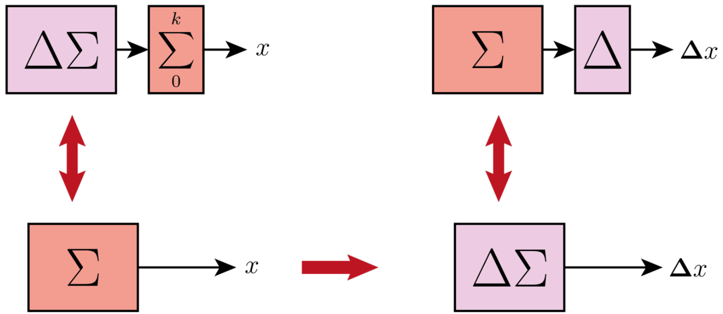


[35] Koelewijn, et. al: Incremental Dissipativity based Control of Discrete-Time Nonlinear Systems using the LPV Framework, *CDC* (2021)
[36] Koelewijn: Analysis and Control of Nonlinear Systems with Stability and Performance Guarantees, *PhD thesis*, 2023

Concept of global stability

- **Krasovskii type of condition**
  - Consider the **velocity** form

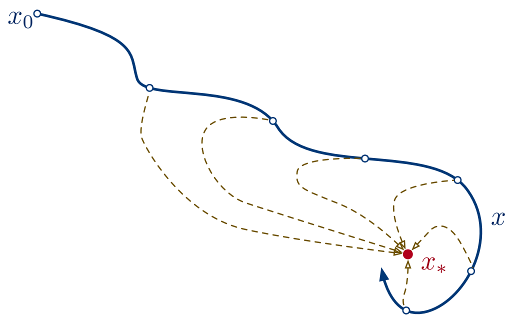
$$\Delta x_{k+1} = \mathcal{A}(x_k, x_{k-1})\Delta x_k$$
$$\Delta x_0 \in \mathbb{R}^{n_x}$$



Shifted Stability
(Asymptotic)

There exists a KL function β such that for any $x_0 \in \mathbb{X}$, there is a $x_* \in \mathbb{X}$ s.t.:

$$\|x_k - x_*\|_2 \leq \underbrace{\beta(\|x_0 - x_*\|_2, k)}_{\kappa e^{-ct}\|x_0 - x_*\|_2}$$

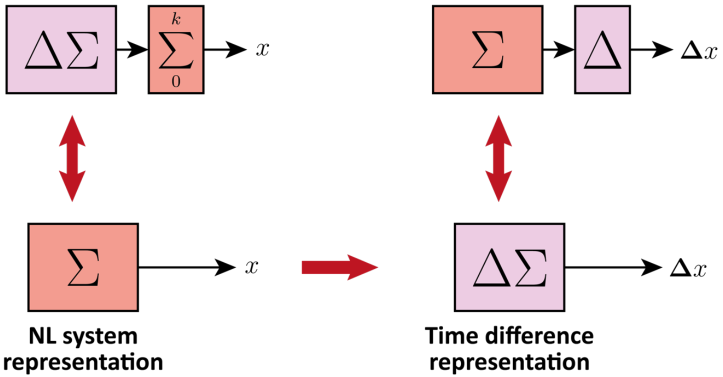


[35] Koelewijn, et. al: Incremental Dissipativity based Control of Discrete-Time Nonlinear Systems using the LPV Framework, *CDC* (2021)
[36] Koelewijn: Analysis and Control of Nonlinear Systems with Stability and Performance Guarantees, *PhD thesis*, 2023

Concept of global stability

- **Krasovskii type of condition**
  - Consider the **velocity** form

$$\Delta x_{k+1} = \mathcal{A}(x_k, x_{k-1})\Delta x_k$$
$$\Delta x_0 \in \mathbb{R}^{n_\mathrm{x}}$$



Shifted Stability
(Asymptotic)

There exists a KL function β such that for any $x_0 \in \mathbb{X}$, there is a $x_* \in \mathbb{X}$ s.t.:

$$\|x_k - x_*\|_2 \leq \underbrace{\beta(\|x_0 - x_*\|_2, k)}_{\kappa e^{-ct}\|x_0 - x_*\|_2}$$

Shifted Stability
(Sufficiency condition)

If there exists a $\mathcal{X} \succ 0$ such that $\forall \bar{x}, \bar{\bar{x}} \in \mathbb{X}$

$$\mathcal{A}^\top(\bar{x}, \bar{\bar{x}})\mathcal{X}\mathcal{A}(\bar{x}, \bar{\bar{x}}) - \mathcal{X} \prec 0$$

[35] Koelewijn, et. al: Incremental Dissipativity based Control of Discrete-Time Nonlinear Systems using the LPV Framework, *CDC* (2021)
[36] Koelewijn: Analysis and Control of Nonlinear Systems with Stability and Performance Guarantees, *PhD thesis*, 2023

Concept of global stability

- **Krasovskii type of condition**
  - Consider the **velocity** form

$$\Delta x_{k+1} = \mathcal{A}(x_k, x_{k-1})\Delta x_k$$
$$\Delta x_0 \in \mathbb{R}^{n_\mathrm{x}}$$

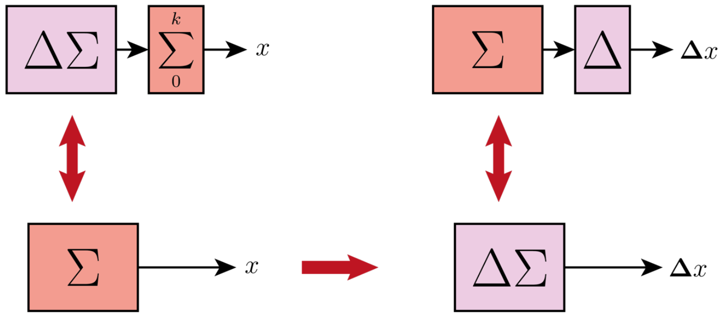
  - Quadratic stability

$$V(x) = \underbrace{(f(x) - x)}_{\Delta x}^\top \mathcal{X} \underbrace{(f(x) - x)}_{\Delta x} \qquad \mathcal{X} \succ 0$$

$$\Delta x^\top \mathcal{A}^\top(qx, x)\mathcal{X}\mathcal{A}(qx, x)\Delta x - \Delta x^\top \mathcal{X} \Delta x \prec 0$$

$$\Downarrow$$

$$(f(x) - x)^\top \mathcal{A}^\top(f(x), x)\mathcal{X}\mathcal{A}(f(x), x)(f(x) - x) - (f(x) - x)^\top \mathcal{X}(f(x) - x) \prec 0$$



NL system representation | **Time difference representation**

Shifted Stability
(Sufficiency condition)

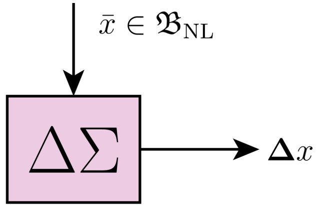
If there exists a $\mathcal{X} \succ 0$ such that $\forall \bar{x}, \bar{\bar{x}} \in \mathbb{X}$

$$\mathcal{A}^\top(\bar{x}, \bar{\bar{x}})\mathcal{X}\mathcal{A}(\bar{x}, \bar{\bar{x}}) - \mathcal{X} \prec 0$$

[35] Koelewijn, et. al: Incremental Dissipativity based Control of Discrete-Time Nonlinear Systems using the LPV Framework, *CDC* (2021)
[36] Koelewijn: Analysis and Control of Nonlinear Systems with Stability and Performance Guarantees, *PhD thesis*, 2023

Concept of global stability

- **Velocity stability**
  - Enough to consider the stability of the velocity form



Velocity system representation

$$\Delta x_{k+1} = \mathcal{A}(\bar{x}_k, \bar{x}_{k-1}) \cdot \Delta x_k$$

Velocity stability

$$\mathcal{X} \succ 0$$

$$\mathcal{A}^\top(\bar{x}, \bar{\bar{x}})\mathcal{X}\mathcal{A}(\bar{x}, \bar{\bar{x}}) - \mathcal{X} \prec 0$$

$$\forall \bar{x}, \bar{\bar{x}} \in \mathbb{X}$$

Shifted stability

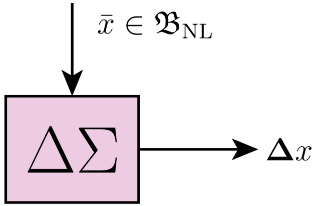
$$V(x) \succ 0$$

$$\Delta V(x) \prec 0$$

$$\forall x \in \mathbb{X}_{x_*} \subseteq \mathbb{X}$$

[35] Koelewijn, et. al: Incremental Dissipativity based Control of Discrete-Time Nonlinear Systems using the LPV Framework, *CDC* (2021)
[36] Koelewijn: Analysis and Control of Nonlinear Systems with Stability and Performance Guarantees, *PhD thesis*, 2023

Concept of global stability



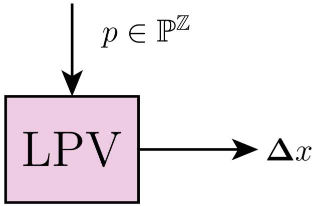
- **Velocity stability**
  - Enough to consider the stability of the velocity form

$$\Delta x_{k+1} = \mathcal{A}(\bar{x}_k, \bar{x}_{k-1}) \cdot \Delta x_k$$

Looks like an LPV form!

[35] Koelewijn, et. al: Incremental Dissipativity based Control of Discrete-Time Nonlinear Systems using the LPV Framework, *CDC* (2021)
[36] Koelewijn: Analysis and Control of Nonlinear Systems with Stability and Performance Guarantees, *PhD thesis*, 2023

Concept of global stability



LPV velocity system representation

- **Velocity stability**
  - Enough to consider the stability of the velocity form

$$\Delta x_{k+1} = A(p_k) \cdot \Delta x_k$$

Looks like an LPV form!

Quadratic LPV stability

$$\mathcal{X} \succ 0$$

$$A^\top(p)\mathcal{X}A(p) - \mathcal{X} \prec 0$$

$$\forall p \in \mathbb{P}$$

Shifted stability

$$V(x) \succ 0$$

$$\Delta V(x) \prec 0$$

$$\forall x \in \mathbb{X}_{x_*} \subseteq \mathbb{X}$$

LPV embedding

We can guarantee stability via an LPV embedding of the velocity form

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[36] Koelewijn: Analysis and Control of Nonlinear Systems with Stability and Performance Guarantees, *PhD thesis*, 2023

Concept of global stability

Theory in pictures

$\Delta\Sigma$ Stability $\rightarrow$ Velocity stability

Stability w.r.t. time-difference dynamics

$\Downarrow$ $\Downarrow$

Σ_* Stability — Shifted stability

Stability w.r.t. arbitrary equilibrium point

$\Downarrow$ $\Downarrow$

Σ Stability $\rightarrow$ Lyapunov stability

Stability w.r.t. the origin

[36] Koelewijn: Analysis and Control of Nonlinear Systems with Stability and Performance Guarantees, *PhD thesis*, 2023
[37] Verhoek, et. al: Convex Incremental Dissipativity Analysis of Nonlinear Systems, *Automatica*, (2023)

Concept of global performance

Theory in pictures

$\Delta\Sigma$

Dissipativity

$\Downarrow$

Σ_*

Dissipativity

$\Downarrow$

Σ

Dissipativity

$\rightarrow$ Velocity dissipativity

Stability w.r.t. time-difference dynamics

$$\Downarrow \quad V(\Delta x_{k_1}) \leq V(\Delta x_{k_2}) + \sum_{t=k_1}^{k_2} s(\Delta u_t, \Delta y_t)$$

Shifted dissipativity*

Dissipativity w.r.t. arbitrary storage point

$$\Downarrow \quad V(x_{k_1}, x_*) \leq V(x_{k_2}, x_*) + \sum_{t=k_1}^{k_2} s(u_t, u_*, y_t, y_*)$$

$\rightarrow$ General dissipativity

Dissipativity w.r.t. the origin

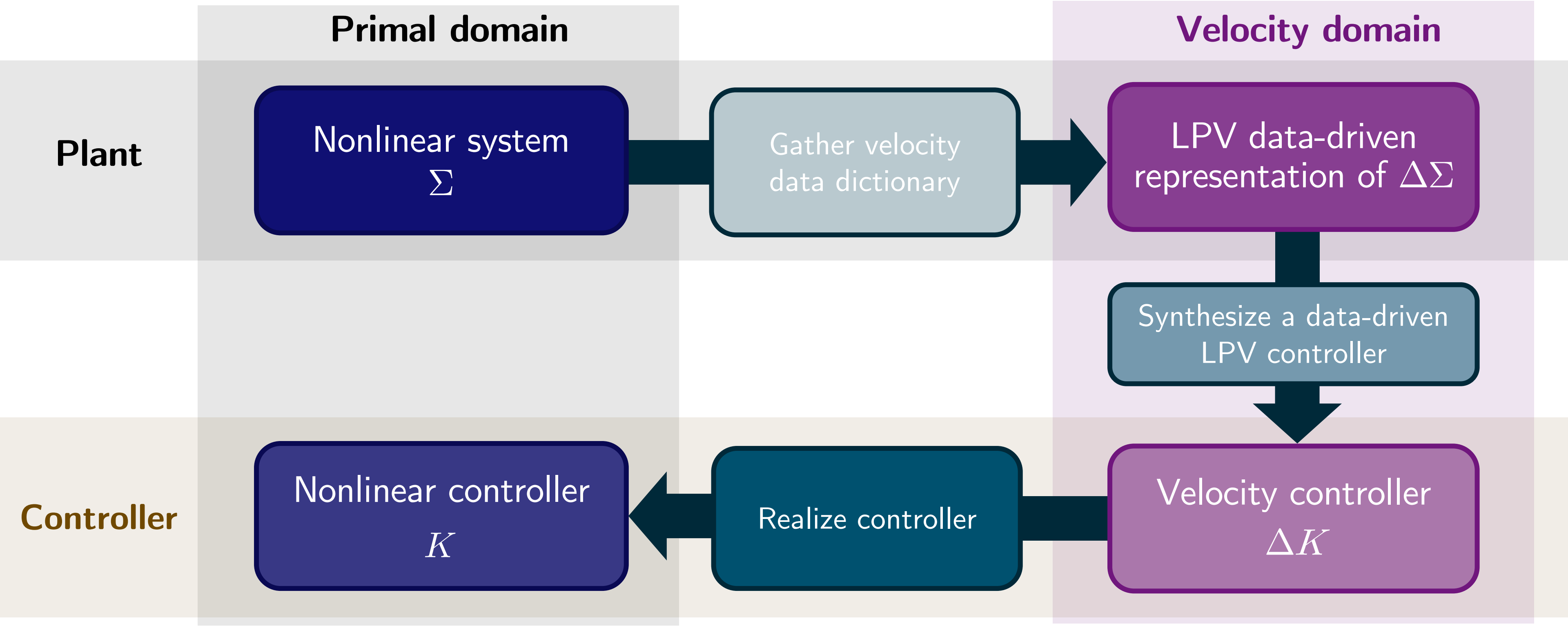
$$V(x_{k_1}) \leq V(x_{k_2}) + \sum_{t=k_1}^{k_2} s(u_t, y_t)$$

[36] Koelewijn: Analysis and Control of Nonlinear Systems with Stability and Performance Guarantees, *PhD thesis*, 2023

[37] Verhoek, et. al: Convex Incremental Dissipativity Analysis of Nonlinear Systems, *Automatica,* (2023)

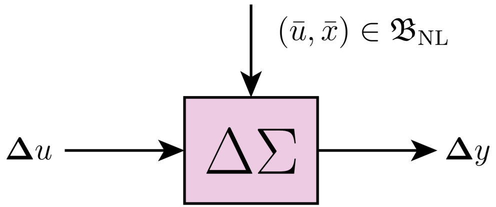
* Under certain conditions, see [Sec. 8.3.3, 36]

Data-driven NL controller synthesis



[38] Verhoek, et. al: Direct data-driven state-feedback control of general nonlinear systems, *CDC,* (2023)

Data-driven NL controller synthesis



Velocity system representation

- **State-feedback synthesis**
  - Consider the time difference form ($w_k = \mathrm{col}(u_k, x_k)$)

$$\begin{aligned} \Delta x_{k+1} &= \mathcal{A}(w_k, w_{k-1})\Delta x_k + \mathcal{B}(w_k, w_{k-1})\Delta u_k, \\ \Delta y_k &= \Delta x_k \end{aligned}$$

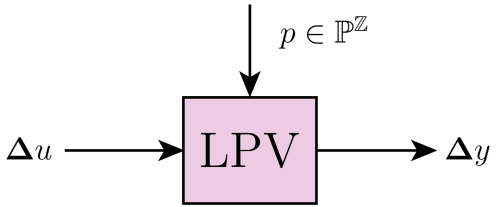
  - Assume a given set of basis functions $\psi_1, \ldots, \psi_{n_\mathrm{p}}$, such that

$$\begin{aligned} \mathcal{A}(w_k, w_{k-1}) &= A_0 + \textstyle\sum_{i=1}^{n_\mathrm{p}} A_i \psi_i(w_k, w_{k-1}) \\ \mathcal{B}(w_k, w_{k-1}) &= B_0 + \textstyle\sum_{i=1}^{n_\mathrm{p}} B_i \psi_i(w_k, w_{k-1}) \end{aligned}$$

(possible to learn these with kernels)

[38] Verhoek, et. al: Direct data-driven state-feedback control of general nonlinear systems, *CDC,* (2023)

Data-driven NL controller synthesis



LPV velocity system representation

- **State-feedback synthesis**
  - LPV embedding

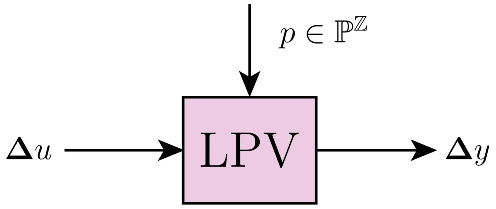
$$\begin{aligned} \Delta x_{k+1} &= A(p_k)\Delta x_k + B(p_k)\Delta u_k, \\ \Delta y_k &= \Delta x_k \end{aligned}$$

  - Scheduling is defined as

$$p_k := \psi(x_k, u_k, x_{k-1}, u_{k-1})$$

[38] Verhoek, et. al: Direct data-driven state-feedback control of general nonlinear systems, *CDC,* (2023)

Data-driven NL controller synthesis



LPV velocity system representation

- **State-feedback synthesis** (velocity domain)

$$\Delta x_{k+1} = A(p_k)\Delta x_k + B(p_k)\Delta u_k,$$
$$\Delta y_k = \Delta x_k$$

Data-dictionary:

$$\mathcal{D}^{\mathrm{NL}}_{N+1} = \{u^{\mathrm{d}}_k, x^{\mathrm{d}}_k\}_{k=0}^{N} \quad \longrightarrow \quad \mathcal{D}^{\Delta}_{N} = \{\Delta u^{\mathrm{d}}_k, \Delta x^{\mathrm{d}}_k, p^{\mathrm{d}}_k\}_{k=1}^{N}$$

$$u^{\mathrm{d}}_k - u^{\mathrm{d}}_{k-1} \qquad x^{\mathrm{d}}_k - x^{\mathrm{d}}_{k-1} \qquad \psi(x^{\mathrm{d}}_k, x^{\mathrm{d}}_{k-1}, u^{\mathrm{d}}_k, u^{\mathrm{d}}_{k-1})$$

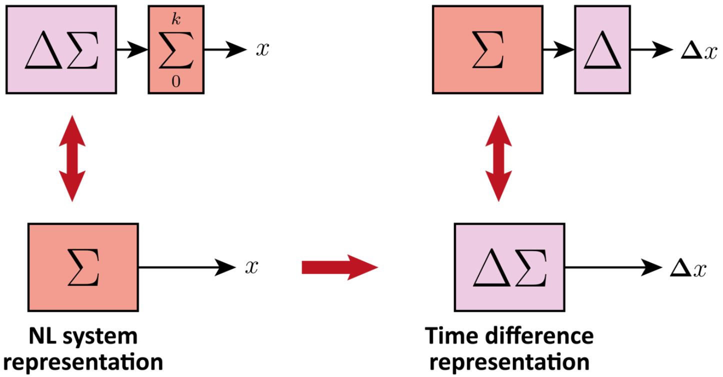
- ➢ Construct data-driven LPV representation (of velocity form)
- ➢ Apply data-driven LPV control synthesis methods from before

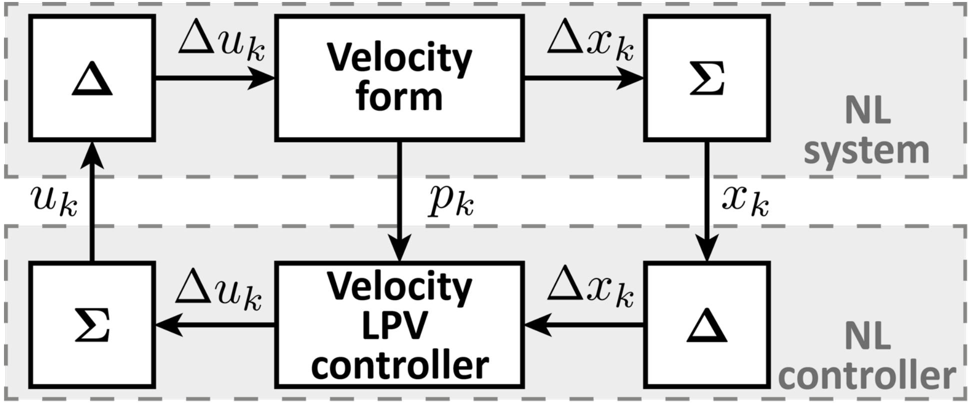
[38] Verhoek, et. al: Direct data-driven state-feedback control of general nonlinear systems, *CDC,* (2023)

Controller realization

- **Controller realization**
  - Velocity form (state-feedback case):

$$\Delta u_k = \Delta K(p_k)\Delta x_k$$
$$= \Delta K(\psi(x_k, u_k, x_{k-1}, u_{k-1}))\Delta x_k$$





[38] Verhoek, et. al: Direct data-driven state-feedback control of general nonlinear systems, *CDC*, (2023)

Controller realization

- **Controller realization**
  - Velocity form (state-feedback case):

$$\begin{aligned}\Delta u_k &= \Delta K(p_k)\Delta x_k \\ &= \Delta K(\psi(x_k, u_k, x_{k-1}, u_{k-1}))\Delta x_k\end{aligned}$$

  - Primal form (realization):

$$K^{\mathrm{NL}} \begin{cases} \chi_{k+1} = \begin{bmatrix} 0 & 0 \\ -\Delta K(p_k) & I \end{bmatrix} \chi_k + \begin{bmatrix} I \\ \Delta K(p_k) \end{bmatrix} x_k \\ u_k = \begin{bmatrix} -\Delta K(p_k) & I \end{bmatrix} \chi_k + \Delta K(p_k) x_k \\ p_k = \psi(x_k, u_k, \chi_k) \\ \chi_k = \begin{bmatrix} x_{k-1}^\top & u_{k-1}^\top \end{bmatrix}^\top \end{cases}$$

Preservation of guarantees

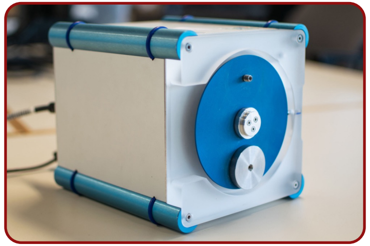
Realization preserves shifted stability & dissipativity!

Achievement:

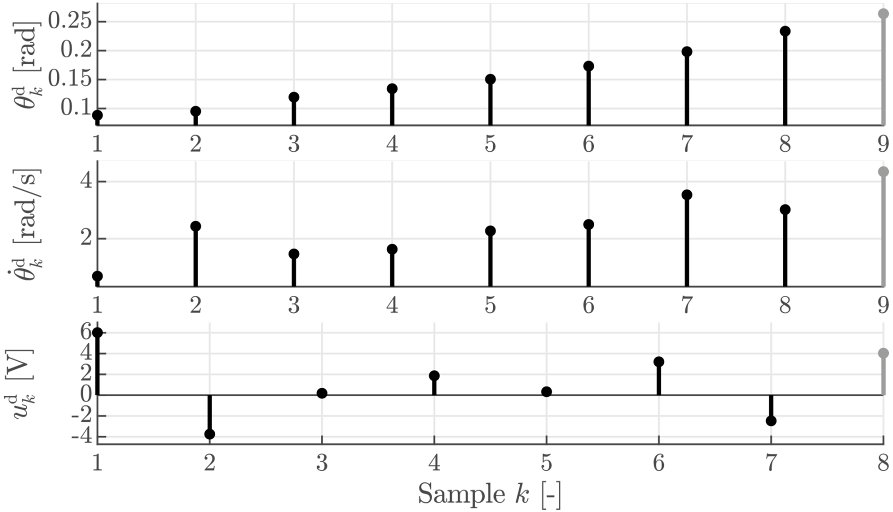
The LPV synthesis is used as a surrogate tool for designing an NL controller with perf. guarantees.

[38] Verhoek, et. al: Direct data-driven state-feedback control of general nonlinear systems, *CDC*, (2023)

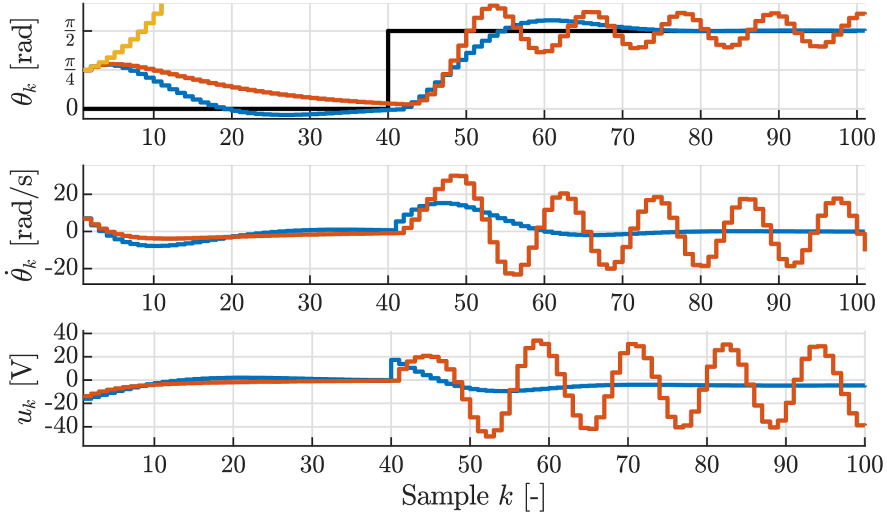
Controller realization



- **Unbalanced disc system (simulation):**
  - Basis functions chosen based on a priori knowledge



Data-dictionary



Data-driven nonlinear, LPV and LTI controller

[38] Verhoek, et. al: Direct data-driven state-feedback control of general nonlinear systems, *CDC*, (2023)

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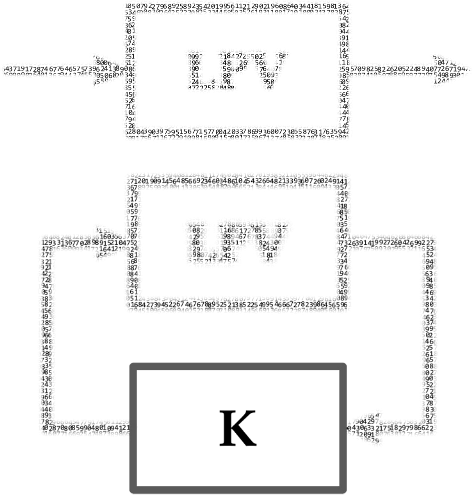


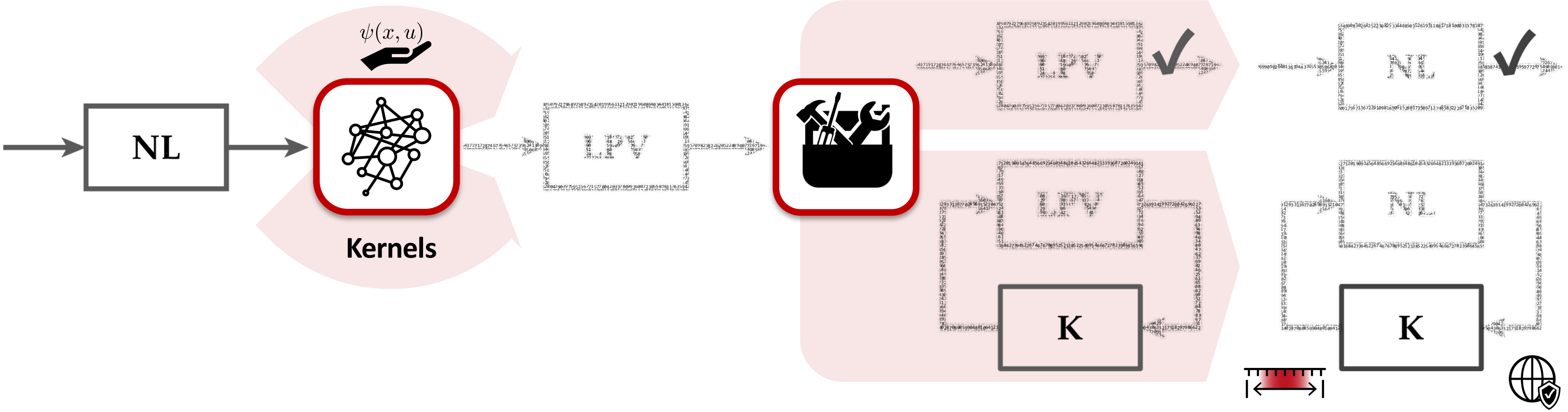
Data-driven LPV representations

Data-driven LPV analysis & control tools

Data-based global guarantees for nonlinear systems

- ~~Learn scheduling map from data~~
- ~~Achieve global guarantees through velocity form~~

Conclusions



A framework for:

Systematic data-driven analysis & control design for nonlinear systems with stability & performance guarantees!

Outlooks

- Scaling up to incremental stability and performance (reference tracking)
- Handling noise and stochastic aspects
- Integration into LPVcore (off-the-shelf software solution)

More info?

My PhD thesis

Recap the talk?

This slide set